

RESONANCE WIDTHS IN A CASE OF MULTIDIMENSIONAL PHASE SPACE TUNNELING

ALAIN GRIGIS¹ & ANDRÉ MARTINEZ²

ABSTRACT. We consider a semiclassical 2×2 matrix Schrödinger operator of the form $P = -h^2 \Delta \mathbf{I}_2 + \text{diag}(x_n - \mu, \tau V_2(x)) + hR(x, hD_x)$, where μ and τ are two small positive constants, V_2 is real-analytic and admits a non degenerate minimum at 0, and $R = (r_{j,k}(x, hD_x))_{1 \leq j,k \leq 2}$ is a symmetric off-diagonal 2×2 matrix of first-order differential operators with analytic coefficients. Then, denoting by e_1 the first eigenvalue of $-\Delta + \langle \tau V_2''(0)x, x \rangle/2$, and under some ellipticity condition on $r_{1,2} = r_{2,1}^*$, we show that, for any μ sufficiently small, and for $0 < \tau \leq \tau(\mu)$ with some $\tau(\mu) > 0$, the unique resonance ρ of P such that $\rho = \tau V_2(0) + (e_1 + r_{2,2}(0, 0))h + \mathcal{O}(h^2)$ (as $h \rightarrow 0_+$) satisfies,

$$\text{Im } \rho = -h^{\frac{3}{2}} f(h, \ln \frac{1}{h}) e^{-2S/h},$$

where $f(h, \ln \frac{1}{h}) \sim \sum_{0 \leq m \leq \ell} f_{\ell,m} h^\ell (\ln \frac{1}{h})^m$ is a symbol with $f_{0,0} > 0$, and S is the imaginary part of the complex action along some convenient closed path containing $(0, 0)$ and consisting of a union of complex null-characteristics of $p_1 := \xi^2 - x_n - \mu$ and $p_2 := \xi^2 + \tau V_2(x)$ (broken instanton). This broken instanton is described in terms of the outgoing and incoming complex Lagrangian manifolds associated with p_2 at the point $(0, 0)$, and their intersections with the characteristic set $p_1^{-1}(0)$ of p_1 .

Keywords: Resonances; Born-Oppenheimer approximation; microlocal tunneling; pseudodifferential operators.

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¹Université Paris 13, Département de Mathématiques, LAGA
UMR CNRS 7539, Av. J.-B. Clément, 93430 Villetaneuse, France,
grigis@math.univ-paris13.fr

²Università di Bologna, Dipartimento di Matematica, Piazza di
Porta San Donato, 40127 Bologna, Italy, martinez@dm.unibo.it
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1. INTRODUCTION

We consider the semiclassical 2×2 matrix Schrödinger operator,

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} + hR(x, hD_x)$$

with,

$$P_1 := -h^2\Delta + x_n - \mu;$$

$$P_2 := -h^2\Delta + \tau V_2(x),$$

where $x = (x', x_n) = (x_1, \dots, x_n)$ is the current variable in \mathbb{R}^n ($n \geq 2$), $h > 0$ denotes the semiclassical parameter, $R(x, hD_x) = (r_{j,k}(x, hD_x))_{1 \leq j, k \leq 2}$ is a formally self-adjoint off-diagonal 2×2 matrix of first-order semiclassical differential operators, and μ, τ are two positive extra-parameters that will be taken sufficiently small later on. Moreover, we assume that the smooth potential V_2 is bounded on \mathbb{R}^n and satisfies,

$$(1.1) \quad V_2 \geq 0; \quad V_2^{-1}(0) = \{0\}; \quad \text{Hess} V_2(0) > 0; \quad \liminf_{|x| \rightarrow \infty} V_2 > 0.$$

In particular, the classically allowed region at energy 0 associated with P_2 (that is, $\{V_2(x) \leq 0\} = \{0\}$) is included into that associated with P_1 (that coincides with $\mathbb{R}^{n-1} \times (-\infty, \mu]$). For this reason, the possible tunneling between the two operators cannot be reduced to a tunneling in the x -variables (as it usually happens, for instance, in scalar multiple-wells problems – see, e.g., [HeSj1, Ma1] – or for shape resonances – see, e.g., [HeSj2]), and actually, such a situation persists under any linear symplectic change of coordinates. Instead, one has to consider the corresponding energy shells which, in our case, are given by $\{(x, \xi) \in \mathbb{R}^{2n}; \xi^2 + x_n = \mu\}$ for P_1 , and by $\{(x, \xi) \in \mathbb{R}^{2n}; \xi^2 + V_2(x) = 0\} = \{(0, 0)\}$ for P_2 . Since they are disjoint, one expects some exponentially small tunneling between them. Indeed, some previous works already exists on such situations (see, e.g., [Ba, Ma2, Ma3, Ma4, Ma5, MaNaSo, NaSo]), but, except in the one dimensional case ([Ba, Na]), no optimal bounds are obtained. Here, we plan to give such an optimal bound in terms of the width of the lower resonance appearing in our model.

Since we plan to study the resonances of P near 0, we also assume that V_2 and the coefficients of R are globally analytic on \mathbb{R}^n , and extend to bounded holomorphic functions in a complex strip of the form $\mathcal{S}_\delta := \{|\text{Im } x| < \delta\}$ for

some $\delta > 0$ (actually, in the sequel, we will need a little bit more). Then, the resonances of P near 0 are defined, e.g., as the eigenvalues close to 0 of the complex translated operator P_θ ($0 < \theta < \delta$) obtained by taking x_n on $\mathbb{R} - i\theta$.

As in [Ma2], one can construct formal WKB eigenfunctions of P_θ with energy close to 0 and, using the ellipticity of $P_{1,\theta}$, one can justify these formal constructions as in [HeSj1], and show that the eigenvalues ρ_j of P_θ in a complex domain of the form $[0, Ch] - i[0, \varepsilon]$ ($\varepsilon > 0$ small enough, $C > 0$ arbitrarily large) admit asymptotic expansions as $h \rightarrow 0_+$, of the form,

$$\rho_j \sim h e_j + h \sum_{k \geq 1} \rho_{j,k} h^{k/2},$$

where $\rho_{j,k} \in \mathbb{R}$ and e_j is the j -th eigenvalue of the harmonic oscillator $-\Delta + \langle V_2''(0)x, x \rangle/2$. Moreover, still as in [Ma2], one can show that the width of ρ_j is exponentially small, that is, there exists $S_j > 0$ such that,

$$|\operatorname{Im} \rho_j| = \mathcal{O}(e^{-S_j/h})$$

uniformly as $h \rightarrow 0$. However, the techniques used in [Ma2] do not permit to have a reasonably good estimate on the best value that one can take for S_j , that is, on the quantity

$$\mathbf{S}_j := -\lim_{h \rightarrow 0} h \ln |\operatorname{Im} \rho_j|.$$

Better estimates can be found in [Ma4], but they are not optimal, as can be deduced from the works [Ba, Na] where optimal results are obtained in the one dimensional case.

The object of this paper is to compute the exact value of \mathbf{S}_1 (when μ is taken sufficiently small and τ is sufficiently small with respect to μ), and show that $\mathbf{S}_1 = S$ where S is the imaginary part of some action integral along the complex bicharacteristics of $p_1(x, \xi) := \xi^2 + x_n - \mu$ and $p_2(x, \xi) := \xi^2 + \tau V_2(x)$ (see (2.2) below). Actually, our result is even more precise, since it gives a complete asymptotic expansion of $e^{S/h} \operatorname{Im} \rho_1$ (see Theorem 3.1). As a by-product of the techniques, we also obtain the estimate $\mathbf{S}_j \geq S$ for all $j \geq 2$ (see Remark 3.2).

Our strategy consists in first making some (global) symplectic change of variables, in order to recover the geometrical situation of [GrMa] (where tunneling was in the position variables only), and then to adapt the argument used in [GrMa]. However, since this symplectic change is quadratic

in the dual variable ξ , a great amount of new technical difficulties appears, due to the bad behavior of the symbols as $|\xi| \rightarrow \infty$. In order to overcome them, it has been necessary for us to considerably improve the estimates of [GrMa] and to introduce some kind of exotic pseudodifferential calculus.

In the next section (Section 2), we describe the geometric objects that enter in the statement of our main result (Section 3). Section 4 is devoted to the reduction of the problem to a geometric situation close to that of [GrMa], and to the construction of a WKB solution. In order to compare this asymptotic solution with the actual resonant state, in Section 5 we develop a generalization of Agmon estimates [Ag], adapted to the peculiarities of the reduced operator (in particular, its somehow exotic aspects from a pseudodifferential point of view). In Section 6, the comparison between asymptotic and actual solution is studied, in the different regions of the x -space. Finally, the proof is completed in Section 7. Besides, various annex results and technical proofs are given in the six appendices.

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2. GEOMETRICAL PRELIMINARIES

This section is devoted to the construction of the broken instanton that is used to express \mathbf{S}_1 as an action integral.

To start with, we work with the principal symbol $p_2(x, \xi) = \xi^2 + \tau V_2(x)$ of P_2 near $(0, 0)$. Since V_2 is analytic, we can consider the Hamilton field $H_{p_2} = (2\xi, -\tau \nabla V_2(x))$ in a complex neighborhood of $(0, 0)$ in \mathbb{C}^{2n} , and, due to the assumptions on V_2 , $(0, 0)$ is a fix point for H_{p_2} . Moreover, following [HeSj1], near $(0, 0)$ we can define the two following (outgoing, respectively incoming) complex Lagrangian submanifolds of $p_2^{-1}(0)$,

$$(2.1) \quad \Sigma_{\pm} = \{(x, \pm i \nabla \varphi_2(x)) ; x \in \mathcal{V}_0\}$$

where $\varphi_2(x) = d_2(x, 0)$, d_2 is the analytic extension of the Agmon distance associated to the degenerate metric $\tau V_2(x) dx^2$ on \mathbb{R}^n , and \mathcal{V}_0 is a complex neighborhood of 0 in \mathbb{C}^n , sufficiently small so that $\varphi_2(x)$ remains holomorphic on it.

Next, we set $p_1(x, \xi) = \xi^2 + x_n - \mu$ and,

$$\Gamma_{\pm} = \Sigma_{\pm} \cap p_1^{-1}(0),$$

and we say that a pair of points $(\rho_+, \rho_-) \in \Gamma_+ \times \Gamma_-$ are in p_1 -correspondence, if there exists $t \in \mathbb{C}$ such that $\rho_- = \exp t H_{p_1}(\rho_+)$ (that is, if ρ_+ and ρ_- are on the same complex integral curve of H_{p_1}).

We have,

Lemma 2.1. *Under Assumption (1.1), there exists a complex neighborhood of $(0, 0)$ that, for each μ sufficiently small and for $\tau \in (0, 1]$, contains a unique pair $(\rho_+, \rho_-) \in \Gamma_+ \times \Gamma_-$ in p_1 -correspondence.*

Proof. By a local change of symplectic coordinates centered at $(0, 0)$ (of the form $(x, \xi) \mapsto (x, \xi) + \mathcal{O}(\xi^2)$), we can reduce to $p_1(x, \xi) = x_n - \mu$ and $p_2(x, \xi) = \xi^2 + \tau V_2(x) + \mathcal{O}(|x|\xi^2 + |\xi|^4)$, both even in ξ . In particular, the image of Σ_{\pm} keeps the same form (2.1) with $\text{Hess} \varphi_2(x) > 0$ near 0 on the real. In these coordinates, H_{p_1} just becomes $-\partial_{\xi_n}$, and thus the points ρ_{\pm} are necessarily given by $\rho_{\pm} = ((x'_{\mu}, \mu); \pm i \nabla \varphi_2(x'_{\mu}, \mu))$, where $x'_{\mu} \in \mathbb{R}^{n-1}$ is the unique solution (near 0) of $\partial_{x'} \varphi_2(x'_{\mu}, \mu) = 0$. \square

Now, let us denote by Γ_2^+ the complex integral curve of H_{p_2} containing ρ_+ , Γ_2^- the complex integral curve of H_{p_2} containing ρ_- , and Γ_1 the complex integral curve of H_{p_1} containing ρ_- and ρ_+ . In particular, $(0, 0) \in \overline{\Gamma_2^{\pm}}$, and we can choose a simple oriented loop $\gamma = \gamma_2^+ \cup \gamma_1 \cup \gamma_2^-$, such that,

γ_2^+ is a path from $(0, 0)$ to ρ_+ in Γ_2^+ ;

γ_1 is a path from ρ_+ to ρ_- in Γ_1 ;

γ_2^- is a path from ρ_- to $(0, 0)$ in Γ_2^- .

Therefore, γ is included in the union of integral curves of H_{p_1} and H_{p_2} , and for this reason we call it a *broken instanton*. Moreover, since the two Hamilton flows define canonical transformations, the action defined by

$$I := \int_{\gamma} \xi dx$$

does not depend on the particular choice of γ , and we set

$$(2.2) \quad S := \text{Im} \int_{\gamma} \xi dx.$$

3. MAIN RESULT

Now, we can state our main result. In addition to the previous assumptions, we also assume that $V_2(x)$ and the coefficients $c(x)$ of R extend to bounded holomorphic function near a complex domain of the form,

$$\tilde{\mathcal{S}}_\delta := \{x \in \mathbb{C}^n ; |\operatorname{Im} x| \leq \delta \langle \operatorname{Re} x_n \rangle^{1/2}\}$$

for some $\delta > 0$. In particular, by Cauchy formula, for any multi-index α , they satisfy,

$$(3.1) \quad |\partial^\alpha V_2(x)| + |\partial^\alpha c(x)| = \mathcal{O}\left(\langle \operatorname{Re} x_n \rangle^{-|\alpha|/2}\right)$$

uniformly on $\tilde{\mathcal{S}}_\delta$. Moreover, we assume that the coefficient of interaction $r_{1,2}$ satisfies,

$$(3.2) \quad r_{1,2}(\rho_+) \neq 0$$

where ρ_+ is defined in Lemma 2.1. Then, our main result is,

Theorem 3.1. *Assume (1.1), (3.1) and (3.2). Then, for any $\mu > 0$ small enough, there exists $\tau(\mu) > 0$ such that, for $\tau \in (0, \tau(\mu)]$, the lowest resonance $\rho_1 = e_1 h + \mathcal{O}(h^2)$ of P is such that,*

$$\operatorname{Im} \rho_1 = -h^{\frac{3}{2}} f\left(h, \ln \frac{1}{h}\right) e^{-2S/h}$$

where $f(h, \ln \frac{1}{h})$ admits an asymptotic expansion of the form,

$$f\left(h, \ln \frac{1}{h}\right) \sim \sum_{0 \leq m \leq \ell} f_{\ell,m} h^\ell \left(\ln \frac{1}{h}\right)^m, \quad (h \rightarrow 0),$$

with $f_{0,0} > 0$, and $S = \operatorname{Im} \int_\gamma \xi dx > 0$ is defined in (2.2).

Remark 3.2. *As in [GrMa], the same techniques also give upper bounds on the widths of the other resonances ρ_j , $j \geq 2$, of the type: $|\operatorname{Im} \rho_j| = \mathcal{O}(h^{-\beta_j} e^{-2S/h})$, for some $\beta_j > 0$. In particular, one has $\mathbf{S}_j \geq S$ for all j .*

Remark 3.3. *The assumption that the matrix $R(x, hD_x)$ is off-diagonal has been made for the sake of simplicity only. Our proof can be adapted without any problem to the case of a general (symmetric) matrix, with the difference that the lowest resonance is now given by: $\rho_1 = (e_1 + r_{2,2}(0,0))h + \mathcal{O}(h^2)$.*

4. REDUCTION AND WKB CONSTRUCTIONS

This section is devoted to the construction of asymptotics solutions of the equation $Pu = \rho u$. In the next sections, these approximate solutions will be shown to be close enough to the actual ones, so that they can be used to find the asymptotics of $\text{Im}\rho$.

First of all, we conjugate our operator P by the unitary Fourier-integral operator,

$$(4.1) \quad \mathcal{U} := \mathcal{F}^{-1} \exp \left[-\frac{2i}{h} \left(\frac{1}{3} \xi_n^3 + \sum_{k=1}^{n-1} \xi_k^2 \xi_n \right) \right] \mathcal{F},$$

where \mathcal{F} stands for the usual h -dependent Fourier transform. The canonical transformation $\kappa : (x, \xi) \mapsto (y, \eta)$ associated with \mathcal{U} is given by,

$$(4.2) \quad \begin{aligned} y_k &= x_k + 4\xi_k \xi_n \quad (k \leq n-1); \\ y_n &= x_n + 2\xi^2; \\ \eta &= \xi, \end{aligned}$$

and thus, under the action of \mathcal{U} , the operator $P(x, hD_x)$ is transformed into $\tilde{P}(y, hD_y) = \text{diag}(\tilde{P}_1, \tilde{P}_2) + h\tilde{R}(y, hD_y)$, with diagonal principal symbol $\tilde{p}(y, \eta) = \text{diag}(\tilde{p}_1(y, \eta), \tilde{p}_2(y, \eta))$, given by,

$$\tilde{p}_j(y, \eta) := p_j(y' - 4\eta_n \eta', y_n - 2\eta^2; \eta).$$

In particular,

$$(4.3) \quad \begin{aligned} \tilde{p}_1(y, \eta) &= -(\eta^2 - y_n + \mu); \\ \tilde{p}_2(y, \eta) &= p_2(y, \eta) + \mathcal{O}(|(y, \eta)|^3); \\ \tilde{p}_2(y, -\eta) &= \tilde{p}_2(y, \eta). \end{aligned}$$

Moreover, thanks to (3.1), we observe that, for any compact set $K \subset \subset \mathbb{R}^n$, the full symbol of \tilde{P} extends, for $x \in K$, to an holomorphic function with respect to η in a complex strip around \mathbb{R}^n , and that its derivatives of any order remain $\mathcal{O}(\langle \eta \rangle^2)$ on this strip.

Then, except from the minus sign in front of \tilde{p}_1 , we see on (4.3) that we are in a situation very similar to that studied by in [GrMa] (see also [Pe]), namely, the transversal crossing between two potentials such that one of them admits a non degenerate point-well, and the two corresponding classically allowed regions are disjoint.

As in [GrMa], the starting point of the construction consists in the WKB asymptotics given near the well $y = 0$ by the method of Helffer and Sjöstrand [HeSj1]. More precisely, because of the matricial nature of the operator and the fact that \tilde{p}_1 is elliptic above $y = 0$, one finds a formal solution w_1 of $\tilde{P}w_1 = \rho w_1$ of the form,

$$(4.4) \quad w_1(y; h) = \begin{pmatrix} ha_1(y, h) \\ a_2(y, h) \end{pmatrix} e^{-\tilde{\varphi}_2(y)/h}$$

where $\tilde{\varphi}_2(y)$ is the (real and non negative on the real domain) solution of $\tilde{p}_2(y, i\nabla\tilde{\varphi}_2(y)) = 0$, $\tilde{\varphi}_2(0) = 0$, defined by,

$$\kappa(\Gamma_+) = \{(y, i\nabla\tilde{\varphi}_2(y)) ; y \text{ close to } 0\}.$$

(In particular, we also have $\nabla\tilde{\varphi}_2(0) = 0$.) Moreover, the two a_j 's ($j = 1, 2$) are classical symbols of order 0 in h , that is, they are formal series in h of the form,

$$(4.5) \quad a_j(y, h) = \sum_{k=0}^{\infty} h^k a_{j,k}(y)$$

with $a_{j,k}$ smooth near 0, and a_2 is elliptic in the sense that $a_{2,0}$ never vanishes.

It is easy to see that the above constructions can be continued along the integral curves of the (real) vector field $\partial_{\eta}\tilde{p}_2(y, i\nabla\varphi_2)D_y = (2\nabla\varphi_2(y) + \mathcal{O}(y^2)).\nabla_y$, as long as $\tilde{p}_1(y, i\nabla\tilde{\varphi}_2(y))$ does not vanish. We set,

$$(4.6) \quad \begin{aligned} \Sigma_2^+ &= \{(y, i\nabla\tilde{\varphi}_2(y)) ; y \in \mathcal{V}\} \\ \Gamma &= \Gamma_{\mu} = \Sigma_2^+ \cap \tilde{p}_1^{-1}(0) \end{aligned}$$

where \mathcal{V} is a fixed (μ -independent) sufficiently small neighbourhood of 0 in \mathbb{C}^n . Then Γ is a complex \mathbb{C} -isotropic submanifold of \mathbb{C}^{2n} (that is, with respect to the complex symplectic form $\sigma = \sum d\eta_k \wedge dy_k$) of dimension $n - 1$, and standard arguments of symplectic geometry show that if \mathcal{T} is a sufficiently small μ -independent neighborhood of 0 in \mathbb{C} , then the set:

$$(4.7) \quad \Sigma_1 = \{\exp tH_{\tilde{p}_1}(y, \eta) ; (y, \eta) \in \Gamma, t \in \mathcal{T}\}$$

is a complex Lagrangian manifold. Since the projection $\Pi_y\Gamma$ of Γ on the base $\{\eta = 0\}$ has an equation of the form $y_n = f(y') = \mu - \delta\mu^2 - f_1(y') + \mathcal{O}(|y'|^3 + \mu^3)$ with $f_1(0) = 0$, $f_1(y') \geq \delta'|y'|^2$ (where $y' = (y_1, \dots, y_{n-1})$ and $\delta, \delta' > 0$ do not depend on μ), an easy computation shows that, for $\mu > 0$ small enough, $\Pi_y H_{\tilde{p}_1}$ keeps transversal to $\Pi_y\Gamma$, as long as $|y'| \ll 1$. Moreover, the only points of Σ_1 where $H_{\tilde{p}_1}$ becomes vertical are above $\{y_n = \mu\}$,

and $\Pi_y \Gamma \cap \mathbb{R}^n \subset \{y_n < \mu\}$. As a consequence, there exists a complex neighborhood \mathcal{W}_μ of Γ such that $\Pi_y \mathcal{W}_\mu \cap \mathbb{R}^n$ is of the form,

$$\Pi_y \mathcal{W}_\mu \cap \mathbb{R}^n = \{-\delta_0 < y_n < g(y'), |y'| < \delta_0\},$$

with $\delta_0 > 0$ independent of μ , and $g(y') = \mu + \mathcal{O}(|y' - y'_\mu|^2)$ (for some $y'_\mu = \mathcal{O}(\mu^2) \in \mathbb{R}^n$), and such that $\Sigma_1 \cap \mathcal{W}_\mu$ projects bijectively on the base. This implies the existence of an analytic function $\tilde{\varphi}_1$ defined on $\Pi_y \mathcal{W}_\mu$, such that,

$$(4.8) \quad \Sigma_1 \cap \mathcal{W}_\mu = \{(y, i\nabla \tilde{\varphi}_1(y)) ; y \in \Pi_y \mathcal{W}_\mu\}.$$

Moreover, $\tilde{\varphi}_1$ can be normalized by requiring that $\tilde{\varphi}_1 = \tilde{\varphi}_2$ on $\Pi_y \Gamma$, so that we then have,

$$(4.9) \quad \tilde{\varphi}_1 = \tilde{\varphi}_2 \quad \text{and} \quad \nabla \tilde{\varphi}_1 = \nabla \tilde{\varphi}_2 \quad \text{on} \quad \Pi_y \Gamma.$$

Using that $\tilde{\varphi}_2$ is real on the real, and $\tilde{p}_1(y, \eta)$ is even with respect to η , we also see that,

$$(4.10) \quad \tilde{\varphi}_1 \quad \text{is real on the real.}$$

Finally, one can easily check that the quantity,

$$S(\mu) := \tilde{\varphi}_1(y'_\mu, \mu),$$

is $\mathcal{O}(\mu^2)$, and that, on $\Pi_y \mathcal{W}_\mu \cap \mathbb{R}^n$, one has $\tilde{\varphi}_1 \leq \tilde{\varphi}_2$. Moreover, since $\tilde{\varphi}_2(y) \geq |y|^2/C$ with some constant $C > 0$, for μ small enough, we have

$$\tilde{\varphi}_2(y) \sim 1 \gg S(\mu) \quad \text{on} \quad \partial(\Pi_y \mathcal{W}_\mu \cap \mathbb{R}^n) \cap \{y_n < g(y')\}.$$

Then, one can deduce,

$$\tilde{\varphi}_1(y) > S(\mu) \quad \text{on} \quad \partial(\Pi_y \mathcal{W}_\mu \cap \mathbb{R}^n) \cap \{f(y') \leq y_n < g(y')\}.$$

On the other hand, one can see as in [HeSj2], Lemma 10.2, that, for $y' \neq y'_\mu$, one has,

$$\tilde{\varphi}_1(y', g(y')) > S(\mu).$$

In addition, a local study of H_{p_1} shows that $(\Pi_y \Sigma_1) \cap \mathbb{R}^n$ has a contact of order exactly 2 with $\{y_n = \mu\} \cap \mathbb{R}^n$ at (y'_μ, μ) , and one can deduce the existence of a constant $C = C(\mu) > 0$ such that $C^{-1}|y' - y'_\mu|^2 \leq \mu - g(y') \leq C|y' - y'_\mu|^2$.

Remark 4.1. *Let us observe that, by standard symplectic arguments, the quantity $S(\mu)$ is nothing but the constant S defined in (2.2).*

As in [GrMa], the extension of the WKB constructions (4.4) accross Γ are obtained by taking a formal ansatz of the form,

$$(4.11) \quad w_2(y; h) = \sum_{k \geq 0} h^k \left(\alpha_k(y, h) Y_{k,0} \left(\frac{z(y)}{\sqrt{h}} \right) + \sqrt{h} \beta_k(y, h) Y_{k,1} \left(\frac{z(y)}{\sqrt{h}} \right) \right) e^{-\psi(y)/h},$$

where,

$$(4.12) \quad \alpha_k(y, h) = \begin{pmatrix} h\alpha_{k,1}(y, h) \\ \alpha_{k,2}(y, h) \end{pmatrix} \quad ; \quad \beta_k(y, h) = \begin{pmatrix} \beta_{k,1}(y, h) \\ h\beta_{k,2}(y, h) \end{pmatrix},$$

$\alpha_{k,j}$ and $\beta_{k,j}$ are formal symbols of the form,

$$(4.13) \quad \sum_{l \geq 0} \sum_{m=0}^l h^l (\ln h)^m \gamma^{l,m}(y)$$

(with $\gamma^{l,m}$ analytic near Γ), the function ψ is defined as,

$$(4.14) \quad \psi(y) := \frac{1}{2} (\tilde{\varphi}_2(y) + \tilde{\varphi}_1(y)),$$

the function $z(y)$ is the only analytic function defined near Γ such that,

$$(4.15) \quad \begin{aligned} z(y)^2 &= 2 (\tilde{\varphi}_2(y) - \tilde{\varphi}_1(y)) \\ z(y) &< 0 \quad \text{on} \quad \Gamma_- := \{y_n < f(y') ; y \in \Pi_y \mathcal{W}_\mu \cap \mathbb{R}^n\}, \end{aligned}$$

and for any $k \geq 0$ and $\varepsilon \in \mathbb{C}$, the function $Y_{k,\varepsilon}$ is the so-called Weber function, defined by,

$$(4.16) \quad Y_{k,\varepsilon}(z) = \partial_\varepsilon^k Y_{0,\varepsilon}(z)$$

where $Y_{0,\varepsilon}$ is the unique entire function with respect to ε and z , solution of the Weber equation,

$$(4.17) \quad Y_{0,\varepsilon}'' + \left(\frac{1}{2} - \varepsilon - \frac{z^2}{4} \right) Y_{0,\varepsilon} = 0$$

such that, for $\varepsilon > 0$, one has,

$$(4.18) \quad Y_{0,\varepsilon}(z) \sim \frac{\sqrt{2\pi}}{\Gamma(\varepsilon)} e^{z^2/4} z^{\varepsilon-1} \quad (z \rightarrow +\infty).$$

Then, a resummation of (4.11) is possible up to an error of order $\mathcal{O}(h^\infty e^{-\phi/h})$, with,

$$(4.19) \quad \begin{aligned} \phi &= \tilde{\varphi}_2 \quad \text{in} \quad \Gamma_- ; \\ \phi &= \tilde{\varphi}_1 = \tilde{\varphi}_2 \quad \text{on} \quad \Gamma ; \\ \phi &= \tilde{\varphi}_1 \quad \text{in} \quad \Gamma_+ := \{y_n > f(y') ; y \in \Pi_y \mathcal{W}_\mu \cap \mathbb{R}^n\}. \end{aligned}$$

Actually, looking a little bit more carefully at the results of [Pe] (in particular Proposition A.2 and the proof of Lemma 3.3), we see that the resummation

can actually be done in a small enough complex neighborhood Ω of Γ , under the condition that $\Omega \subset \{y \in \mathbb{C}^n ; |\operatorname{Im} z(y)| \leq C\sqrt{h}\}$ (where $C > 0$ is an arbitrary constant). In that case, the resummation is valid up to $\mathcal{O}(h^\infty e^{-\operatorname{Re} \phi/h})$ error-terms. In what follows, we will use the notations,

$$\psi_\delta(y) := \psi(y + i\delta) \quad ; \quad \phi_\delta(y) := \phi(y + i\delta),$$

where $\delta = \delta(h) = \mathcal{O}(\sqrt{h}) \in \mathbb{R}^n$.

For $\nu_0 > 0$ and $g \in C^\infty(\mathbb{R}^{2n} ; \mathbb{R}_+)$, we denote by $S_{\nu_0}(g(x, \xi))$ the set of (possibly h -dependent) functions $p \in C^\infty(\mathbb{R}^{2n})$ that extend to holomorphic functions with respect to ξ in the strip,

$$\mathcal{A}_{\nu_0} := \{(x, \xi) \in \mathbb{R}^n \times \mathbb{C}^n ; |\operatorname{Im} \xi| < \nu_0\},$$

and such that, for all $\alpha \in \mathbb{N}^{2n}$, one has,

$$(4.20) \quad \partial^\alpha p(x, \xi) = \mathcal{O}(g(x, \operatorname{Re} \xi)),$$

uniformly with respect to $(x, \xi) \in \mathcal{A}_{\nu_0}$ and $h > 0$ small enough. We also denote by $S_0(g)$ the analogous space of smooth symbols obtained by substituting \mathbb{R}^{2n} to \mathcal{A}_{ν_0} .

If a and b are (scalar) formal symbols of the type (4.13), and $k \in \mathbb{N}$, we set,

$$(4.21) \quad I_k(a, b)(y ; h) = a(y ; h)Y_{k,0}\left(\frac{z(y)}{\sqrt{h}}\right) + b(y ; h)Y_{k,1}\left(\frac{z(y)}{\sqrt{h}}\right).$$

For $M \in \mathbb{Z}$ and $\Omega \subset \mathbb{R}^n$ open, we also consider the space of sequences of formal symbols,

$$S^M(\Omega) := \{a = (a_k)_{k \in \mathbb{N}} ; a_k(y, h) = \sum_{l=-M}^{\infty} \sum_{m=0}^l h^l (\ln h)^m \gamma_k^{l,m}(y) ; \gamma_k^{l,m} \in C^\infty(\Omega)\}.$$

and, for $a, b \in S^M(\Pi_y \mathcal{W}_\mu)$, we set,

$$(4.22) \quad I(a, b)e^{-\psi_\delta/h} := \sum_{k \geq 0} h^k I_k(a_k, \sqrt{h}b_k)e^{-\psi_\delta/h}.$$

Then, the arguments of [GrMa], Section 4, give a natural way to make act $\operatorname{Op}_h^W(p)$ on expressions of the type,

$$(4.23) \quad w = \begin{pmatrix} I(h\alpha_1, \beta_1) \\ I(\alpha_2, h\beta_2) \end{pmatrix} e^{-\psi_\delta/h},$$

where $\alpha_j = (\alpha_{j,k})_{k \geq 0}$ and $\beta_j = (\beta_{j,k})_{k \geq 0}$ are in $S^0(\Pi_y \mathcal{W}_\mu)$ ($j = 1, 2$). Moreover, this action is well defined up to error terms that are exponentially smaller than $e^{\operatorname{Re} \phi_\delta/h}$.

Now, denote by p the full symbol of P , and recall the definition $\tilde{P} := \mathcal{U}P\mathcal{U}^{-1}$, where \mathcal{U} is given in (4.1). In order to be able to make act \tilde{P} , too, on such expressions, we first show,

Lemma 4.2. *Let $c = c(x)$ satisfying (3.1). Then, for any $\chi = \chi(y_n) \in C_0^\infty(\mathbb{R})$, there exists $\nu_0 > 0$ such that the operator $\chi\mathcal{U}c\mathcal{U}^{-1}$ can be written as,*

$$\chi\mathcal{U}c\mathcal{U}^{-1} = \text{Op}_h^W(\check{c} + r),$$

with $\check{c} \in S_{\nu_0}(1)$, $r \in S_0(1)$, and,

$$r = \mathcal{O}(e^{-\nu_0/h}) \text{ in } S_0(1).$$

Moreover, the symbol \check{c} satisfies,

$$\check{c}(y, \eta) = \chi(y_n)c(y' - 4\eta_n\eta', y_n - 2\eta^2) + \mathcal{O}(h) \text{ in } S_{\nu_0}(1).$$

Proof. A straightforward computation leads to,

$$(4.24) \quad \chi\mathcal{U}c\mathcal{U}^{-1} = \text{Op}_h^L(a),$$

with,

$$\begin{aligned} a(y, \eta) &= \frac{1}{(2\pi h)^n} \int \chi(y_n) e^{i(y-x)(\xi-\eta)/h} \times \\ &\quad \times c(x' - 2\xi_n(\xi' + \eta'), x_n - \frac{2}{3}(\eta_n^2 + \eta_n\xi_n + \xi_n^2) - 2|\eta'|^2) dx d\xi. \end{aligned}$$

In this integral, we can perform the change of contour of integration,

$$\mathbb{R}^n \ni x \mapsto x - i\delta \frac{\xi - \eta}{\langle \xi - \eta \rangle} \in \mathbb{C}^n,$$

where $\delta > 0$ is a small enough constant. Setting $f = (f', f_n) := \delta(\xi - \eta)/\langle \xi - \eta \rangle$ and,

$$F(x, \eta, \xi) := (x' - if' - 2\xi_n(\xi' + \eta'), x_n - if_n - \frac{2}{3}(\eta_n^2 + \eta_n\xi_n + \xi_n^2) - 2|\eta'|^2),$$

we obtain,

$$\begin{aligned} a(y, \eta) &= \frac{1}{(2\pi h)^n} \int \chi(y_n) e^{i(y-x)(\xi-\eta)/h - \frac{\delta}{h}|\xi-\eta|^2/\langle \xi-\eta \rangle} c(F(x, \eta, \xi)) dx d\xi \\ &= \frac{1}{(2\pi h)^n} \int \chi(y_n) e^{i(y-x)\xi/h - \frac{\delta}{h}|\xi|^2/\langle \xi \rangle} c(F(x, \eta, \xi + \eta)) dx d\xi, \end{aligned}$$

and thus, for any fix $\chi_0 \in C_0^\infty(\mathbb{R}^n; [0, 1])$ such that $\chi_0 = 1$ near 0,

$$\begin{aligned} a(y, \eta) &= \frac{1}{(2\pi h)^n} \int \chi(y_n) \chi_0(\xi) e^{i(y-x)\xi/h - \frac{\delta}{h}|\xi|^2/\langle \xi \rangle} c(F(x, \eta, \xi + \eta)) dx d\xi \\ &\quad + r_1(y, \eta), \end{aligned}$$

where, by means of integrations by parts in ξ , the symbol r_1 can be seen to satisfy,

$$\partial^\alpha r_1 = \mathcal{O}(e^{-\delta'/h}) \text{ in } S_0(1),$$

for some $\delta' > 0$ independent of μ and h .

Then, we split the previous integral into,

$$I_1 + I_2 + I_3 := \int_{|x_n| \leq 2C} + \int_{2C \leq |x_n| \leq 2C + \langle \eta \rangle^2} + \int_{|x_n| \geq 2C + \langle \eta \rangle^2},$$

where $C > 0$ is chosen sufficiently large in order to have $|y_n - x_n| \geq |x_n|/2$ when $|x_n| \geq 2C$ and $y_n \in \text{Supp } \chi$.

On $\{|x_n| \leq 2C + \langle \eta \rangle^2\} \cap \text{Supp } \chi_0$, we observe that $|\text{Re } F_n| \sim \langle \eta \rangle^2$. Therefore, if $\chi_1 \in C_0^\infty(\mathbb{R}^n; [0, 1])$ is such that $\chi_1 = 1$ near 0 and $\text{Supp } \chi_1 \subset \{\chi_0 = 1\}$, in this region we can perform the change of contour of integration,

$$\mathbb{R}^n \ni \xi \mapsto \tilde{\xi} := \xi + i\delta' \chi_1(\xi) \frac{y - x}{\langle y - x \rangle},$$

that gives us,

$$\begin{aligned} I_1 &= \chi(y_n) \int_{|x_n| \leq 2C} \mathcal{O}(\chi_0(\xi) h^{-n} e^{-\frac{\delta' \chi_1(\xi) |y-x|^2}{2h \langle y-x \rangle} - \frac{\delta' |\xi|^2}{h \langle \xi \rangle}}) c(F(x, \eta, \tilde{\xi} + \eta)) dx d\xi; \\ I_2 &= \chi(y_n) \int_{2C \leq |x_n| \leq 2C + \langle \eta \rangle^2} \mathcal{O}(\chi_0(\xi) h^{-n} e^{-\frac{\delta' \chi_1(\xi) |y-x|^2}{2h \langle y-x \rangle} - \frac{\delta' |\xi|^2}{h \langle \xi \rangle}}) \\ &\quad \times c(F(x, \eta, \tilde{\xi} + \eta)) dx d\xi, \end{aligned}$$

and thus, using Assumption (3.1), we see that, for $\nu_0 > 0$ sufficiently small, we have $I_1 \in S_{\nu_0}(1)$. Moreover, a stationary-phase expansion shows that,

$$I_1 = \chi(y_n) c(F(y, \eta, \eta)) + \mathcal{O}(h) = \chi(y_n) c(y' - 4\eta_n \eta', y_n - 2\eta^2) + \mathcal{O}(h),$$

where the estimates $\mathcal{O}(h)$ hold in the space $S_{\nu_0}(1)$.

On the other hand, since $|x - y| \geq |x_n - y_n| \geq |x_n|/2$ for $|x_n| \geq 2C$ and $y_n \in \text{Supp } \chi$, we see that,

$$I_2 = \mathcal{O}(e^{-\nu_1/h}) \text{ in } S_0(1),$$

for some $\nu_1 > 0$.

Finally, in the expression of I_3 , we perform the change of contour of integration,

$$\mathbb{R}^n \ni \xi \mapsto \hat{\xi} := \xi + i\delta \chi_1(\xi) \frac{y - x}{\langle y - x \rangle \langle \eta \rangle},$$

and since, in the relevant region, $|y - x| \geq |x_n - y_n| \geq |x_n|/2 \geq C + \langle \eta \rangle^2/2$, this leads to,

$$I_3 = \chi(y_n) \int_{|x_n| \geq 2C + \langle \eta \rangle^2} \mathcal{O}(\chi_0(\xi) h^{-n} e^{-\frac{\delta' \chi_1 |x|}{h \langle \eta \rangle} - \frac{\delta' \chi_1 \langle \eta \rangle}{h} - \frac{\delta' |\xi|^2}{h \langle \xi \rangle}}) \\ \times V_2(F(x, \eta, \hat{\xi} + \eta)) dx d\xi,$$

and thus,

$$I_3 = \mathcal{O}(e^{-\delta'/h}) \text{ in } S_0(1).$$

Therefore, we have proved the result of the lemma, except from the fact that we have worked with the left-quantization instead of the Weyl-one. But the passage from one quantization to the other one can be done in a standard way (see, e.g., [Ma6]), and this completes the proof of the lemma. \square

As a corollary, and since $\mathcal{U} h D_x \mathcal{U}^{-1} = h D_y$, we immediately obtain (denoting by $\mathcal{M}_2(E)$ the space of 2×2 matrices with coefficients in E),

Proposition 4.3. *Assume (3.1), and denote by T_δ the complex translation given by $T_\delta u(y) := u(y + i\delta)$. Then, for any $\chi = \chi(y_n) \in C_0^\infty(\mathbb{R})$, there exists $\nu_0 > 0$ (independent of μ) such that the operator $\chi \tilde{P}^\delta := \chi T_\delta \tilde{P} T_\delta^{-1}$ can be written as,*

$$\chi \tilde{P}^\delta = \text{Op}_h^W(\check{p}_\delta + r_0),$$

with $\check{p}_\delta \in \mathcal{M}_2(S_{\nu_0}(\langle \eta \rangle^2))$, $r_0 \in \mathcal{M}_2(S_0(\langle \eta \rangle^2))$, and,

$$r_0 = \mathcal{O}(e^{-\nu_0/h}) \text{ in } \mathcal{M}_2(S_0(\langle \eta \rangle^2)).$$

Moreover, the symbol \check{p}_δ satisfies,

$$\check{p}_\delta(y, \eta) = \chi(y_n) p(y' + i\delta' - 4\eta_n \eta', y_n + i\delta_n - 2\eta^2; \eta) + \mathcal{O}(h) \text{ in } \mathcal{M}_2(S_{\nu_0}(\langle \eta \rangle^2)).$$

Consequently, for μ sufficiently small, and up to error terms that are exponentially smaller than $e^{-\text{Re } \phi_\delta/h}$, the operator \tilde{P}^δ naturally acts on expressions of the type (4.23).

Then, proceeding as in [GrMa] (but in a quite simpler geometric situation, here), we finally obtain the existence of a μ -independent neighborhood \mathcal{V} of 0 in \mathbb{R}^n , such that, if one sets,

$$(4.25) \quad \begin{aligned} \mathcal{V}_\mu^- &= \mathcal{V} \cap \{y_n < f(y')\}; \\ \mathcal{V}_\mu^+ &= \mathcal{V} \cap \{f(y') < y_n < g(y')\}, \end{aligned}$$

then, we have,

Proposition 4.4. *There exists a neighborhood Ω_μ of $\Pi_y\Gamma$ and symbols,*

$$(4.26) \quad \begin{aligned} a_1, a_2 &\in S^0(\mathcal{V}_\mu^-), \quad a_2 \text{ elliptic}; \\ \alpha_1, \alpha_2, \beta_1, \beta_2 &\in S^0(\Omega_\mu), \quad \alpha_2 \text{ and } \beta_1 \text{ elliptic}; \\ b_1, b_2 &\in S^0(\mathcal{V}_\mu^+), \quad b_1 \text{ elliptic}, \end{aligned}$$

such that, if one sets,

$$\begin{aligned} w_1 &= \begin{pmatrix} ha_1 \\ a_2 \end{pmatrix} e^{-\tilde{\varphi}_2/h} \quad ; \quad w_2 = \begin{pmatrix} I(h\alpha_1, \beta_1) \\ I(\alpha_2, h\beta_2) \end{pmatrix} e^{-\psi/h} \quad ; \\ w_3 &= \sqrt{2\pi h} \begin{pmatrix} b_1 \\ hb_2 \end{pmatrix} e^{-\tilde{\varphi}_1/h}, \end{aligned}$$

then, one formally has,

$$\tilde{P}w_1 = \rho w_1 \quad ; \quad \tilde{P}w_2 = \rho w_2 \quad ; \quad \tilde{P}w_3 = \rho w_3.$$

Moreover, if \tilde{w}_1 , \tilde{w}_2 and \tilde{w}_3 denote resummations of w_1 , w_2 and w_3 respectively, one has,

$$(4.27) \quad \tilde{w}_1 - \tilde{w}_2 = \mathcal{O}(h^\infty e^{-\tilde{\varphi}_2/h}) \quad \text{in } \mathcal{V}_\mu^- \cap \Omega_\mu$$

$$(4.28) \quad \tilde{w}_2 - \tilde{w}_3 = \mathcal{O}(h^\infty e^{-\tilde{\varphi}_1/h}) \quad \text{in } \mathcal{V}_\mu^+ \cap \Omega_\mu.$$

Now, let (χ_1, χ_2, χ_3) be a partition of unity adapted to $(\mathcal{V}_\mu^-, \Omega_\mu, \tilde{\mathcal{V}}_\mu^+)$ with $\tilde{\mathcal{V}}_\mu^+ \subset \subset \mathcal{V}_\mu^+$ and $\chi_3 \in C_0^\infty(\mathcal{V}_\mu^+)$, and set,

$$(4.29) \quad \tilde{U} = \sum_{j=1}^3 \chi_j \tilde{w}_j \in C_0^\infty(\mathcal{V} \cap \{y_n < g(y')\}).$$

Then, the arguments in [GrMa] show that,

$$(4.30) \quad \tilde{P}\tilde{U} = \rho\tilde{U} + \mathcal{O}(h^\infty e^{-\phi/h}) \quad \text{uniformly in } \{\chi_1 + \chi_2 + \chi_3 = 1\}.$$

Finally, the caustic set $\mathcal{C} := \{y_n = g(y')\}$ can be crossed over, too, by using, as in [HeSj2] Section 10, an Airy representation near \mathcal{C} , and by using the stationary phase method in $\mathcal{V} \cap \{y_n > g(y')\}$ in order to recover there an expression of the type $\sqrt{2\pi h} \begin{pmatrix} c_1 \\ hc_2 \end{pmatrix} e^{\tilde{\varphi}/h}$ where the symbol c_1 is elliptic and $\tilde{\varphi}$ is a complex-valued analytic function that satisfies $(\nabla \tilde{\varphi})^2 = \mu - y_n$, $\text{Re } \tilde{\varphi} = Cte = S(\mu)$ on $\gamma_\mu := \{\exp tH_{\tilde{p}_1}(y'_\mu, \mu); t \in \mathbb{R} \text{ small enough}\}$, $\text{Re } \tilde{\varphi}(y) \geq S(\mu) + \delta_1 \text{dist}(y, \gamma_\mu)^2$ for some $\delta_1 > 0$. In particular, one also has,

$$(4.31) \quad |\partial_{y_n} \text{Im } \tilde{\varphi}(y'_\mu, y_n)| = \sqrt{y_n - \mu}.$$

By using an additional cut-off function χ_4 near $(0, \mu)$, and by shrinking a little bit (but still in a μ -independent way) \mathcal{V} around 0, we finally obtain a

function,

$$\mathbf{w} = \sum_{j=1}^4 \chi_j \tilde{w}_j \in C_0^\infty(\mathcal{V} \cap \{y_n < g(y') + 2\varepsilon\})$$

($\varepsilon = \varepsilon(\mu) > 0$), such that, setting $\tilde{\mathcal{V}}_1 := \{\chi_1 + \chi_2 + \chi_3 + \chi_4 = 1\}$ and $\tilde{\mathcal{V}}_2 := \bigcup_{1 \leq j \leq 4} \text{Supp}(\chi_j)$, one has,

$$(4.32) \quad \tilde{P}\mathbf{w} = \rho\mathbf{w} + \mathcal{O}(h^\infty e^{-\varphi/h}) \quad \text{on} \quad \tilde{\mathcal{V}}_1 \cap \{y_n < g(y') + \varepsilon\};$$

$$(4.33) \quad \tilde{P}\mathbf{w} = \rho\mathbf{w} + \mathcal{O}(h^{-N_0} e^{-S(\mu)/h}) \quad \text{on} \quad \tilde{\mathcal{V}}_2 \cap \{y_n \geq g(y')\};$$

$$(4.34) \quad \tilde{P}\mathbf{w} = \rho\mathbf{w} + \mathcal{O}(h^{-N_0} e^{-\varphi/h}) \quad \text{on} \quad \tilde{\mathcal{V}}_2 \setminus \tilde{\mathcal{V}}_1,$$

with $\varphi = \tilde{\varphi}_2$ in $\tilde{\mathcal{V}}_2 \cap \{y_n \leq f(y')\}$, $\varphi = \tilde{\varphi}_1$ in $\tilde{\mathcal{V}}_2 \cap \{f(y') \leq y_n \leq g(y')\}$, $\varphi = \text{Re } \tilde{\varphi}$ in $\tilde{\mathcal{V}}_2 \cap \{g(y') \leq y_n \leq g(y') + \varepsilon\}$, and $N_0 \geq 0$.

Observe that, by construction, $\varphi > S(\mu)$ on $(\tilde{\mathcal{V}}_2 \setminus \tilde{\mathcal{V}}_1) \cap \{y_n \leq g(y')\}$.

5. AGMON ESTIMATES

In order to estimate the width of the resonance, we need to prove that the asymptotic solution \mathbf{w} constructed in the previous section gives a sufficiently good approximation of the true resonant state \mathbf{u} , specially in the region $\{y_n > \mu\}$.

For this, we would like to adopt the usual strategy of Agmon estimates (see [Ag]), but, since we work with operators that are not differential (even at the level of principal symbols), we have to be more careful. In particular, our weight functions need to be smooth (and not only Lipschitz like in the case of a Schrödinger operator). Moreover, these weight functions need to be very close (at least up to $O(h)$) to the $C^{1,1}$ weight ϕ introduced in the previous section. This lead to estimates of the type $\partial^\alpha \tilde{\phi} = O(h^{-(|\alpha|-2)_+})$ for such functions, and also estimates of the type $\partial_x^\alpha \partial_\xi^\beta a(x, \xi) = O(h^{-(|\alpha|-1)_+})$ for the related symbols.

Thus, we first have to establish some preliminary results for such objects.

5.1. Preliminaries. In the sequel, we denote by λ_h a positive function of h such that,

$$\lambda_h \geq 1 \quad ; \quad h\lambda_h = \mathcal{O}(1) \quad \text{as } h \rightarrow 0_+.$$

For $\Omega \subset \mathbb{R}^n$, we also denote by $\mathbf{1}_\Omega$ the characteristic function of Ω .

Proposition 5.1. *Let $\nu_0 > 0$, $m \geq 0$, $a = a(y, \eta) \in S_{\nu_0}(\langle \eta \rangle^{2m})$, and $\Omega \subset \mathbb{R}^n$. For $h > 0$ small enough, let also $\Phi_h \in C^\infty(\mathbb{R}^n)$ real valued, such that,*

$$(5.1) \quad \sup |\nabla \Phi_h| < \nu_0,$$

and, for any multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq 2$,

$$(5.2) \quad \partial^\alpha \Phi_h(y) = \mathcal{O}\left((1 + \lambda_h \mathbf{1}_\Omega(y))h^{2-|\alpha|}\right),$$

uniformly for $y \in \mathbb{R}^n$ and $h > 0$ small enough. Then, for any $\tilde{\Omega} \subset \mathbb{R}^n$ with $\text{dist}(\Omega, \mathbb{R}^n \setminus \tilde{\Omega}) > 0$, the operator $e^{\Phi_h/h} A e^{-\Phi_h/h} := e^{\Phi_h/h} \text{Op}_h^W(a) e^{-\Phi_h/h}$ can be split into ,

$$(5.3) \quad e^{\Phi_h/h} A e^{-\Phi_h/h} = \text{Op}_h^W(a_0(y, \eta + i\nabla \Phi_h(y))) + hB + R$$

where a_0 is the principal symbol of $A = \text{Op}_h^W(a)$, and the two operators B and R satisfy,

$$(5.4) \quad |\langle Bu, u \rangle| \leq C_1 \left(\|\langle hD_y \rangle^m u\|^2 + \lambda_h \|\langle hD_y \rangle^m u\|_{L^2(\tilde{\Omega})}^2 \right);$$

$$(5.5) \quad |\langle Ru, u \rangle| \leq C_2 e^{(2 \sup |\Phi| - \varepsilon_0)/h} \|\langle hD_y \rangle^m u\|^2,$$

where the two positive constants C_2 and ε_0 do not depend on Φ , and where the estimate holds for all $h > 0$ small enough and $u \in H^m(\mathbb{R}^n)$.

Proof. For $u \in C_0^\infty(\mathbb{R}^n)$, we write,

$$e^{\Phi/h} A e^{-\Phi/h} u(y) = \frac{1}{(2\pi h)^n} \int e^{i(y-y')\eta/h + (\Phi(y) - \Phi(y'))/h} a\left(\frac{y+y'}{2}, \eta\right) u(y') dy' d\eta,$$

and we first make the change of contour of integration given by,

$$\mathbb{R}^n \ni \eta \mapsto \eta + i\delta \frac{y-y'}{\langle y-y' \rangle} \in \mathbb{C}^n,$$

where $\delta > 0$ is fixed small enough (independently of Φ).

Then, we fix a cut-off $\chi_1 \in C_0^\infty(\mathbb{R}^n; [0, 1])$ such that $\chi_0 = 1$ near 0. Using this cut-off to split the integral, and setting $f := \delta \frac{y-y'}{\langle y-y' \rangle}$ and $g := \delta \frac{|y-y'|^2}{\langle y-y' \rangle}$, we obtain,

$$e^{\Phi/h} A e^{-\Phi/h} u(y) = I_1 + I_2,$$

with,

$$I_1 := \frac{1}{(2\pi h)^n} \int \chi_0(y-y') e^{i(y-y')\eta/h + (\Phi(y) - \Phi(y'))/h - g/h} \\ \times a\left(\frac{y+y'}{2}, \eta + if\right) u(y') dy' d\eta;$$

$$I_2 := \frac{1}{(2\pi h)^n} \int (1 - \chi_0(y - y')) e^{i(y-y')\eta/h + (\Phi(y) - \Phi(y'))/h - g/h} \\ \times a\left(\frac{y + y'}{2}, \eta + if\right) u(y') dy' d\eta.$$

Now, on the support of $1 - \chi_0(y - y')$, we have $g \geq 2\varepsilon_0$ for some $\varepsilon_0 > 0$ that does not depend on Φ , and thus, by the Calderón-Vaillancourt theorem (see, e.g., [Ma6]), we immediately obtain,

$$(5.6) \quad |\langle I_2, u \rangle| \leq C e^{(2 \sup |\Phi| - \varepsilon_0)/h} \|\langle h D_y \rangle^m u\|^2,$$

where the constant $C > 0$ does not depend on Φ .

On the other hand, if δ has been chosen sufficiently small, the property (5.1) shows that, in I_1 , we can make the change of contour of integration given by,

$$\mathbb{R}^n \ni \eta \mapsto \eta + i\Psi(y, y'),$$

where the smooth vectorial function $\Psi(y, y')$ is defined by the identity: $\Phi(y) - \Phi(y') = (y - y')\Psi(y, y')$. Then, denoting by Op_h the semiclassical quantization of symbols depending on $3n$ variables (see e.g. [Ma6] Section 2.5), we obtain,

$$I_1 = \text{Op}_h \left(e^{-g/h} \chi_0(y - y') a\left(\frac{y + y'}{2}, \eta + i\Psi(y, y') + if\right) \right) u(y).$$

Next, passing to symbols depending on $2n$ variables (see e.g. [Ma6] Theorem 2.7.1), this gives,

$$I_1 = \text{Op}_h^W(b) u(y)$$

with b given by the oscillatory integral,

$$b(y, \eta) = \frac{1}{(2\pi h)^n} \int e^{i(\eta' - \eta)\theta/h - g_1/h} \chi_0(\theta) a\left(y, \eta' + i\Psi\left(y + \frac{\theta}{2}, y - \frac{\theta}{2}\right) + if_1\right) d\eta' d\theta,$$

where we have used the notations: $g_1 := \delta|\theta|^2/\langle\theta\rangle$; $f_1 := \delta\theta/\langle\theta\rangle$. Then, we need,

Lemma 5.2. *For any $\delta > 0$, the symbol b can be split into,*

$$b(y, \eta) = a(y, \eta + i\nabla\Phi(y)) + \tilde{b}(y, \eta),$$

where, for any $\alpha, \beta \in \mathbb{N}^n$, \tilde{b} satisfies,

$$\partial_y^\alpha \partial_\eta^\beta \tilde{b}(y, \eta) = \mathcal{O} \left((1 + \lambda_h \mathbf{1}_{\Omega_\delta}(y)) h^{1-|\alpha|} \langle \eta \rangle^{2m} \right),$$

uniformly for $h > 0$ small enough and $(y, \eta) \in \mathbb{R}^n \times \mathbb{C}^n$, $|\text{Im } \eta|$ small enough.

Here, we have set,

$$\Omega_\delta := \{x \in \mathbb{R}^n; \text{dist}(x, \Omega) \leq \delta\}.$$

Proof. see Appendix A. \square

This means that $h^{-1}\langle\xi\rangle^{-2m}\tilde{b}$ belongs to the space of symbols $\tilde{S}_0(\lambda_h, \Omega_\delta)$ introduced in Appendix B. Therefore, applying Proposition B.1 (iv), we deduce that, for any $\delta > 0$, one has,

$$(5.7) \quad |\langle \text{Op}_h^W(\tilde{b})u, u \rangle| = \mathcal{O}(h\lambda_h) \| \langle hD_y \rangle^m u \|_{L^2(\Omega_\delta)}^2 + \mathcal{O}(h) \| \langle hD_y \rangle^m u \|^2,$$

uniformly as $h \rightarrow 0$ (see e.g. [Ma6] Exercise 2.10.15). Moreover, one also has,

$$a(y, \eta + i\nabla\Phi_h(y)) = a_0(y, \eta + i\nabla\Phi_h(y)) + ha_1(y, \eta + i\nabla\Phi_h(y)),$$

and,

$$\begin{aligned} \partial_y^\alpha \partial_\eta^\beta [a_1(y, \eta + i\nabla\Phi_h(y))] &= \mathcal{O} \left((1 + \sup_{1 \leq |\gamma| \leq |\alpha|+1} |\partial^\gamma \Phi_h|) \langle \eta \rangle^{2m} \right) \\ &= \mathcal{O}((1 + \lambda_h h^{1-|\alpha|}) \langle \eta \rangle^{2m}) \\ &= \mathcal{O}(h^{-|\alpha|} \langle \eta \rangle^{2m}). \end{aligned}$$

Thus, $|\langle \text{Op}_h^W(a_1)u, u \rangle| = \mathcal{O}(1) \| \langle hD_y \rangle^m u \|^2$, and the result follows. \square

Thanks to the previous lemma, only the principal symbols of our operators will be involved in the estimates. However, the study of \tilde{P}_2 remains delicate since its principal symbol is not a polynome in η . However, due to the parity of $\tilde{p}_2(y, \eta)$ with respect to η , we have for any real-valued Lipschitz function $\Phi(y)$ with $|\nabla\Phi|$ small enough,

$$(5.8) \quad \text{Re } \tilde{p}_2(y, \eta + i\nabla\Phi(y)) = \tilde{p}_2(y, i\nabla\Phi(y)) + \frac{1}{2} \langle \text{Hess}_\eta \tilde{p}_2(y, i\nabla\Phi(y)) \eta, \eta \rangle + \mathcal{O}(|\eta|^3)$$

and therefore, since $\text{Hess}_\eta \tilde{p}_2(0, 0)$ is positive definite,

$$(5.9) \quad \text{Re } \tilde{p}_2(y, \eta + i\nabla\Phi(y)) \geq \tilde{p}_2(y, i\nabla\Phi(y)) + \delta_0 |\eta|^2$$

for some constant $\delta_0 > 0$ and (y, η) small enough. Actually, we have a more complete result on $\tilde{P}_2 := \mathcal{U}P_2\mathcal{U}^{-1}$, as stated in the following proposition:

Proposition 5.3. *Let $K \subset \mathbb{R}$ be a fix compact set, and let $\Phi = \Phi_{h,\mu} \in C^\infty(\mathbb{R}^n; \mathbb{R})$ be such that,*

$$(5.10) \quad \text{Supp} \nabla\Phi \subset \{y_n \in K\},$$

and,

$$(5.11) \quad |\Phi| + |\nabla\Phi| = \mathcal{O}(\mu)$$

uniformly with respect to $h, \mu > 0$ both small enough. Moreover, assume that for some $\Omega \subset \mathbb{R}^n$, for any multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq 2$, and for any fix $\mu > 0$ small enough, one has,

$$(5.12) \quad \partial^\alpha \Phi = \mathcal{O} \left((1 + \lambda_h \mathbf{1}_\Omega) h^{2-|\alpha|} \right),$$

uniformly with respect to $h > 0$ small enough.

Then, for μ sufficiently small, there exist three positive constants $\tau(\mu)$, $\delta = \delta(\mu)$ and C_μ , such that, for all $\tau \in (0, \tau(\mu)]$ and for all $h > 0$ small enough, one has,

$$\operatorname{Re} e^{\Phi/h} \tilde{P}_2 e^{-\Phi/h} \geq \frac{1}{C_\mu} \left(-\frac{h^2}{2} \Delta + \tilde{p}_2(y, i\nabla \Phi(y)) \right) - C_\mu h (1 + \lambda_h \mathbf{1}_{\Omega_\delta}).$$

Proof. Let $\chi \in C_0^\infty(\mathbb{R})$ be such that $\chi = 1$ in a neighborhood of $K \cup \{0\}$. Since $P_2 = -h^2 \Delta + \tau V_2$ with V_2 satisfying (3.1), by Lemma 4.2 there exists $\nu_1 > 0$, such that,

$$\chi(y_n) \tilde{P}_2 = \chi(y_n) (-h^2 \Delta_y) + \tau \operatorname{Op}_h^W(\check{V}_2) + \mathcal{O}(e^{-\nu_1/h}),$$

with $\check{V}_2 \in S_{\nu_1}(1)$ satisfying,

$$\check{V}_2(y, \eta) = \chi(y_n) V_2(y' - 4\eta_n \eta', y_n - 2\eta^2) + \mathcal{O}(h) \text{ in } S_{\nu_1}(1).$$

Moreover, by assumption $|\Phi| + |\nabla \Phi(y)| \leq C\mu$ for some constant $C > 0$, and thus, for μ sufficiently small, we can apply Proposition 5.1 with $m = 0$, and we obtain,

$$e^{\Phi/h} \operatorname{Op}_h^W(\check{V}_2) e^{-\Phi/h} = \operatorname{Op}_h^W \left(\chi(y_n) \check{V}_2(y, \eta + i\nabla \Phi) \right) + hB + R,$$

where we have set,

$$(5.13) \quad \check{V}_2(y, \eta) := V_2(y' - 4\eta_n \eta', y_n - 2\eta^2),$$

and with,

$$|\langle Bu, u \rangle| \leq C_1 (\|u\|^2 + \lambda_h \|u\|_{L^2(\Omega_\delta)}^2); \quad |\langle Ru, u \rangle| \leq C_2 e^{(2C\mu - \varepsilon_0)/h} \|u\|^2.$$

Here, $\varepsilon_0 > 0$ does not depend on μ , and thus, for μ small enough, this gives,

$$\operatorname{Re} e^{\Phi/h} \operatorname{Op}_h^W(\check{V}_2) e^{-\Phi/h} \geq \operatorname{Re} \operatorname{Op}_h^W \left(\chi(y_n) \check{V}_2(y, \eta + i\nabla \Phi) \right) - Ch(1 + \lambda_h \mathbf{1}_{\Omega_\delta}).$$

Since we also have,

$$\operatorname{Re} e^{\Phi/h} \chi(y_n) (-h^2 \Delta) e^{-\Phi/h} = \operatorname{Re} \chi(y_n) (-h^2 \Delta - |\nabla \Phi|^2) + \mathcal{O}(h),$$

we obtain,

$$(5.14) \quad \begin{aligned} & \operatorname{Re} e^{\Phi/h} \chi \tilde{P}_2 e^{-\Phi/h} \\ & \geq \operatorname{Re} \operatorname{Op}_h^W \left(\chi(y_n) (\eta^2 - |\nabla \Phi|^2 + \tau \tilde{V}_2(y, \eta + i\nabla \Phi)) \right) \\ & \quad - Ch(1 + \lambda_h \mathbf{1}_{\Omega_\delta}), \end{aligned}$$

with some new constant $C > 0$.

On the other hand, using (5.13) and making a Taylor expansion, we see,

$$\begin{aligned} \operatorname{Re} \tilde{V}_2(y, \eta + i\nabla \Phi) &= V_2(y' - 4\eta_n \eta' + 4(\partial_{y_n} \Phi) \nabla_{y'} \Phi, y_n - 2\eta^2 + 2(\nabla \Phi)^2) \\ &\quad - \langle Y, AY \rangle, \end{aligned}$$

where we have set,

$$Y = (Y', Y_n) := (4\eta_n \nabla_{y'} \Phi + \eta' (\partial_{y_n} \Phi), 4\eta \cdot \nabla \Phi) \in \mathbb{R}^n,$$

and,

$$A := \int_0^1 \operatorname{Re} V_2''(y' - 4\eta_n \eta' + 4(\partial_{y_n} \Phi) \nabla_{y'} \Phi - itY', y_n - 2\eta^2 + 2(\nabla \Phi)^2 - itY_n) dt.$$

In particular, by (5.11)-(5.12), we see that, for μ sufficiently small,

$$\chi(y_n) \langle Y, AY \rangle \in \mathcal{S}_0(\lambda_h, \Omega),$$

where $\mathcal{S}_0(\lambda_h, \Omega)$ is the exotic class of symbols introduced in Appendix B. Actually, both Y and χA are in similar (vectorial) classes of symbols (with weight $\langle \eta \rangle$ for Y), and, for any $u \in C_0^\infty(\mathbb{R}^n)$, the results of Appendix B permit us to write,

$$\begin{aligned} \langle \operatorname{Op}_h^W(\chi(y_n) \langle Y, AY \rangle) u, u \rangle &= \langle \operatorname{Op}_h^W(\chi A) \operatorname{Op}_h^W(Y) u, \operatorname{Op}_h^W(Y) u \rangle \\ &\quad + \mathcal{O}_\mu(h\lambda_h) \|\langle hD_x \rangle u\|_{\Omega_\delta}^2 + \mathcal{O}_\mu(h) \|\langle hD_x \rangle u\|^2 \\ &= \mathcal{O}_\mu(\|\operatorname{Op}_h^W(Y) u\|^2 \\ &\quad + \mathcal{O}_\mu(h\lambda_h) \|\langle hD_x \rangle u\|_{\Omega_\delta}^2 + \mathcal{O}_\mu(h) \|\langle hD_x \rangle u\|^2 \\ &= \mathcal{O}_\mu(\|h\nabla u\|^2 + h\lambda_h \|u\|_{\Omega_\delta}^2 + h\|u\|^2). \end{aligned}$$

Here, \mathcal{O}_μ means that the estimate is not necessarily uniform with respect to μ (and this is precisely the reason why we need the small parameter τ in our problem).

In the same way, we also have,

$$\begin{aligned} & \operatorname{Op}_h^W(\chi(y_n) V_2(y' - 4\eta_n \eta' + 4(\partial_{x_n} \Phi) \nabla_{x'} \Phi, y_n - 2\eta^2 + 2(\nabla \Phi)^2)) \\ &= \operatorname{Op}_h^W(\chi(y_n) V_2(y' + 4(\partial_{x_n} \Phi) \nabla_{x'} \Phi, y_n + 2(\nabla \Phi)^2)) + B_1, \end{aligned}$$

with,

$$\langle B_1 u, u \rangle = \mathcal{O}_\mu(\|h \nabla u\|^2 + h \lambda_h \|u\|_{\Omega_\delta}^2 + h \|u\|^2).$$

Therefore, coming back to (5.14), we deduce,

$$\begin{aligned} e^{\Phi/h} \chi(y_n) \tilde{P}_2 e^{-\Phi/h} &\geq \text{Op}_h^W((\chi(y_n) - C_\mu \tau) \eta^2 + \chi(y_n) \tilde{p}_2(y, i \nabla \Phi(y))) \\ (5.15) \quad &\quad - C_\mu h \lambda_h \mathbf{1}_{\Omega_\delta} - C_\mu h, \end{aligned}$$

where the constant $C_\mu > 0$ may depend on μ . Since $e^{\Phi/h}(1 - \chi) \tilde{P}_2 \chi e^{-\Phi/h} = (e^{-\Phi/h} \chi \tilde{P}_2 (1 - \chi) e^{\Phi/h})^*$, we also have,

$$\begin{aligned} \text{Re } e^{\Phi/h} (1 - \chi) \tilde{P}_2 \chi e^{-\Phi/h} &\geq \text{Op}_h^W((1 - \chi(y_n) \chi(y_n) (\eta^2 + \tilde{p}_2(y, i \nabla \Phi(y))) \\ (5.16) \quad &\quad - C_\mu \tau (h D_y)^2 - C_\mu h \lambda_h \mathbf{1}_{\Omega_\delta} - C_\mu h. \end{aligned}$$

Finally, since Φ is constant on the support of $(1 - \chi)$, we can write,

$$\begin{aligned} (5.17) \quad &\text{Re } \langle e^{\Phi/h} (1 - \chi) \tilde{P}_2 (1 - \chi) e^{-\Phi/h} u, u \rangle \\ &= \langle (1 - \chi) \tilde{P}_2 (1 - \chi) u, u \rangle \\ &= \langle \text{Op}_h^W((1 - \chi)^2 \eta^2) u, u \rangle + \tau \langle (1 - \chi) \mathcal{U} V_2 \mathcal{U}^{-1} (1 - \chi) u, u \rangle. \end{aligned}$$

Here, one must be aware of the fact that the pseudodifferential operator $(1 - \chi) \mathcal{U} V_2 \mathcal{U}^{-1} (1 - \chi)$, though being uniformly bounded on L^2 , does not have its symbol in the class $S_0(1; \mathbb{R}^n)$ (there are directions where the repeated derivatives with respect to η increase more and more as $|\eta| \rightarrow \infty$). However, we have,

Lemma 5.4. *The operator $(1 - \chi) \mathcal{U} V_2 \mathcal{U}^{-1} (1 - \chi)$ satisfies,*

$$\langle (1 - \chi) \mathcal{U} V_2 \mathcal{U}^{-1} (1 - \chi) u, u \rangle \geq \frac{(1 - \chi)^2}{C} - C(h D_y)^2 - C h^2,$$

where $C > 0$ is a constant.

Proof. Let $\chi_0 \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ be such that $\text{Supp } \chi_0 \subset \{\chi(y_n) = 1\}$ and $\chi_0(0) \neq 0$. Then, the assumptions on V_2 imply the existence of a positive constant C such that,

$$V_2 + \chi_0^2 \geq \frac{1}{C}.$$

As a consequence,

$$(5.18) \quad (1 - \chi) \mathcal{U} V_2 \mathcal{U}^{-1} (1 - \chi) \geq \frac{(1 - \chi)^2}{C} - (1 - \chi) \mathcal{U} \chi_0^2 \mathcal{U}^{-1} (1 - \chi),$$

and thus, it is enough to estimate $(1 - \chi) \mathcal{U} \chi_0^2 \mathcal{U}^{-1} (1 - \chi)$. We write it as,

$$(1 - \chi) \mathcal{U} \chi_0^2 \mathcal{U}^{-1} (1 - \chi) = \mathcal{U} A \chi_0^2 A \mathcal{U}^{-1},$$

with,

$$A := \mathcal{U}^{-1}(1 - \chi)\mathcal{U} = 1 - \mathcal{U}^{-1}\chi\mathcal{U}.$$

By a straightforward computation (similar to that of (4.24)), we have,

$$A\chi_0 = \chi_0 - \text{Op}_h^R(a),$$

with,

$$a(y, \eta) := \frac{1}{2\pi h} \int e^{i(x_n - y_n)(\xi_n - \eta_n)/h} \chi(x_n + \theta(\xi_n, \eta)) \chi_0(y) dx_n d\xi_n,$$

where we have set: $\theta(\xi, \eta) = 2|\eta'|^2 + 2(\eta_n^2 + \eta_n \xi_n + \xi_n^2)/3$. Therefore, writing $\chi(x_n + \theta) = \chi(x_n) + \theta \int_0^1 \chi'(x_n + t\theta) dt$ and using the fact that $\chi\chi_0 = \chi_0$, we find,

$$\chi_0(y) - a(y, \eta) = \text{Op}_h^R(b),$$

with,

$$b(y, \eta) := \frac{1}{2\pi h} \int_0^1 \int_{\mathbb{R}^2} e^{i(x_n - y_n)(\xi_n - \eta_n)/h} \theta(\xi_n, \eta) \chi'(x_n + t\theta(\xi, \eta)) \chi_0(y) dx d\xi dt.$$

Then, writing $\theta(\xi_n, \eta) = 2\eta^2 + 2\eta_n(\xi_n - \eta_n) + \frac{2}{3}(\xi_n - \eta_n)^2$, and making integrations by parts with respect to x_n and ξ_n , for any $M, N \geq 1$, we find,

$$b(y, \eta) = 2\eta^2 A_1 + 2h\eta_n A_2 + h^2 A_3,$$

where, for $j = 1, 2, 3$,

$$A_j = \int_0^1 \int_{\mathcal{E}_t} \mathcal{O} \left(h^{-1} \frac{(1 + t|\xi_n| + t|\eta|)^M}{\langle h^{-1}(\xi_n - \eta_n) \rangle^N \langle x_n - y_n \rangle^M} \right) dx_n d\xi_n dt.$$

Here, \mathcal{E}_t stands for the support of $\chi(x_n + t\theta(\xi_n, \eta))\chi_0(y)$. In particular, on this set we have,

$$t|\xi_n| + t|\eta| = \mathcal{O}(\sqrt{t|\xi_n|^2 + t|\eta|^2}) = \mathcal{O}(\langle x_n \rangle^{1/2} + \langle \xi_n - \eta_n \rangle) \quad ; \quad |y| = \mathcal{O}(1),$$

and thus, for $N \geq 2M$, this gives,

$$h^{-1} \frac{(1 + t|\xi_n| + t|\eta|)^M}{\langle h^{-1}(\xi_n - \eta_n) \rangle^N \langle x_n - y_n \rangle^M} = \mathcal{O} \left(\frac{h^{-1}}{\langle h^{-1}(\xi_n - \eta_n) \rangle^M \langle x_n \rangle^{M/2}} \right).$$

Therefore, $A_j = \mathcal{O}(1)$ uniformly, and the same is true for all its derivatives.

The same procedure also shows that $a \in S_0(1)$, and we can write,

$$\begin{aligned} \langle A\chi_0^2 Au, u \rangle &= \langle \text{Op}_h^R(b)\chi_0 Au, u \rangle = \mathcal{O}(\|hD_y \chi_0 Au\| \cdot \|hD_y u\| \\ &\quad + h\|\chi_0 Au\| \cdot \|hD_y u\| + h^2\|\chi_0 Au\| \cdot \|u\|), \end{aligned}$$

that is, since $\chi_0 A = \text{Op}_h^L(a)$ and $[D_y, \chi_0 A]$ are uniformly bounded,

$$\langle A\chi_0^2 Au, u \rangle = \mathcal{O}(\|hD_y u\|^2 + h^2\|u\|^2).$$

Then, by (5.18) (and the fact that D_y and \mathcal{U} commute), the result follows. \square

Going back to (5.17), we deduce,

$$\begin{aligned} & \operatorname{Re} e^{\Phi/h} (1 - \chi) \tilde{P}_2 (1 - \chi) e^{-\Phi/h} \\ & \geq \operatorname{Op}_h^W \left(((1 - \chi)^2 - C\tau)\eta^2 + \frac{\tau}{C}(1 - \chi)^2 - Ch^2 \right). \end{aligned}$$

Summing up with (5.15)-(5.16), we have proved,

$$\begin{aligned} (5.19) \quad \operatorname{Re} e^{\Phi/h} \tilde{P}_2 e^{-\Phi/h} & \geq \operatorname{Op}_h^W \left((1 - C\tau)\eta^2 + (1 - (1 - \chi)^2) \tilde{p}_2(y, i\nabla\Phi(y)) \right) \\ & \quad + \frac{\tau}{C} \operatorname{Op}_h^W \left((1 - \chi)^2 \right) - Ch\lambda_h \mathbf{1}_{\Omega_\delta} - Ch. \end{aligned}$$

On the other hand, on the support of $(1 - \chi)$, we have $\tilde{p}_2(y, i\nabla\Phi(y)) = \tilde{p}_2(y, 0) = \tau V_2(y) \geq 0$, and thus $(1 - (1 - \chi)^2) \tilde{p}_2(y, i\nabla\Phi(y)) + \frac{\tau}{C} \geq \frac{\tau}{C} \geq C_1^{-1} \tau V_2(y) = C_1^{-1} \tilde{p}_2(y, i\nabla\Phi(y))$, for some positive constant C_1 . As a consequence, for $\tau > 0$ sufficiently small, we deduce from (5.19),

$$\operatorname{Re} e^{\Phi/h} \tilde{P}_2 e^{-\Phi/h} \geq \frac{1}{C} \operatorname{Op}_h^W \left(\eta^2 + \tilde{p}_2(y, i\nabla\Phi(y)) \right) - Ch\lambda_h \mathbf{1}_{\Omega_\delta} - Ch,$$

where the (new) constant $C > 0$ may depend on μ . Thus, Proposition 5.3 is proved. \square

5.2. Agmon Estimates. Now, we can specify what we call Agmon estimates in our case. In the rest of the paper, we set,

$$k := h \ln h^{-1},$$

and, for $t \in \mathbb{R}$, we consider the translation $e^{itD_{x_n}}$ given by,

$$e^{itD_{x_n}} f(y) = f(y', y_n + t).$$

We also set,

$$\tilde{P}^t := e^{itD_{y_n}} \tilde{P} e^{-itD_{y_n}}.$$

Since $e^{itD_{y_n}}$ and \mathcal{U} commute, we have,

$$\tilde{P}^t = \mathcal{U} P^t \mathcal{U}^{-1},$$

where,

$$P^t := e^{itD_{x_n}} P e^{-itD_{x_n}} = \begin{pmatrix} P_1^t & 0 \\ 0 & P_2^t \end{pmatrix} + hR(x', x_n + t; hD_x),$$

with,

$$P_1^t := -h^2 \Delta + x_n + t - \mu \quad ; \quad P_2^t := -h^2 \Delta + \tau V_2(x', x_n + t).$$

In particular, by Assumption (3.1), we see that \tilde{P}^t extends analytically to complex values of t , $|t|$ small enough, and we set,

$$(5.20) \quad Q := \tilde{P}^{-ik}.$$

Then, the essential spectrum of Q is $\mathbb{R} - ik$, and the resonances of P (or \tilde{P}) close enough to 0 are discrete eigenvalues of Q . In particular, the resonant state $\tilde{\mathbf{u}} = \mathcal{U}\mathbf{u}$ of \tilde{P} associated to the resonance ρ (where \mathbf{u} is the corresponding resonant state of P), is such that $\mathbf{v} := e^{kD_{y_n}}\tilde{\mathbf{u}}$ is well defined, belongs to $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$, and satisfies,

$$(Q - \rho)\mathbf{v} = 0.$$

Moreover, \mathbf{v} can be normalized by requiring: $\|\mathbf{v}\| = 1$.

In the sequel, we set,

$$Q_j = \tilde{P}_j^{-ik} := \mathcal{U}P_j^{-ik}\mathcal{U}^{-1}.$$

Actually, since $\mathcal{F}x_n\mathcal{F}^{-1} = ih\partial_{\xi_n}$, the operator Q_1 can be explicitly computed, and one finds,

$$Q_1 = h^2\Delta + y_n - ik - \mu.$$

Moreover, setting $\tilde{R}_k := \mathcal{U}R(x', x_n - ik; hD_x)\mathcal{U}^{-1}$, we have,

$$(5.21) \quad Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} + h\tilde{R}_k,$$

Now, we fix an h -dependent cut-off function $\chi_h \in C^\infty(\mathbb{R}; [0, 1])$, such that,

$$\chi_h = 1 \text{ on } (-\infty, -2k^{2/3}]; \chi_h = 0 \text{ on } [-k^{2/3}, +\infty); \partial^\alpha \chi_h = \mathcal{O}(k^{-2|\alpha|/3}),$$

and we set,

$$(5.22) \quad \begin{aligned} \tilde{V}_1(y_n) &:= (\mu - y_n)\chi_h(y_n - \mu) + k^{2/3}\langle y_n \rangle(1 - \chi_h(y_n - \mu)); \\ \tilde{Q}_1 &:= h^2\Delta - ik - \tilde{V}_1(y_n); \\ \tilde{Q} &= \begin{pmatrix} \tilde{Q}_1 & 0 \\ 0 & Q_2 \end{pmatrix} + h\tilde{R}_k. \end{aligned}$$

We also denote by $\mathcal{D}(\tilde{Q}_1)$ the (natural) domain of \tilde{Q}_1 , and we prove,

Proposition 5.5. *Let Φ be as in Proposition 5.3. Then, for any $\delta > 0$, there exists a positive constant $C_2 = C_2(\mu)$ such that, for any $v = (v_1, v_2) \in$*

$\mathcal{D}(\tilde{Q}_1) \times H^2(\mathbb{R}^n)$, and setting $\tilde{v} := (-v_1, v_2)$, one has,

$$\begin{aligned} \operatorname{Re} \langle e^{\Phi/h} \tilde{Q}v, e^{\Phi/h} \tilde{v} \rangle &\geq \frac{1}{C_2} \|h \nabla(e^{\Phi/h} v_1)\|^2 + \|h \nabla(e^{\Phi/h} v_2)\|^2 \\ &\quad + \frac{1}{C_2} \langle \tilde{p}_2(y, i \nabla \Phi(y)) e^{\Phi/h} v_2, e^{\Phi/h} v_2 \rangle \\ &\quad + \langle (\tilde{V}_1(y_n) - (\nabla \Phi)^2) e^{\Phi/h} v_1, e^{\Phi/h} v_1 \rangle \\ &\quad - C_2 h \|e^{\Phi/h} v\|^2 - C_2 h \lambda_h \|e^{\Phi/h} v_2\|_{\Omega_\delta}^2, \end{aligned}$$

uniformly for $h > 0$ small enough.

Proof. We first prove,

Lemma 5.6. *Let $c = c(x)$ satisfying (3.1). Then, with Φ as in Proposition 5.3, the operator $e^{\Phi/h} \mathcal{U} \mathcal{M}^{-1} e^{-\Phi/h}$ is uniformly bounded on L^2 as $h \rightarrow 0_+$.*

Proof. Using Lemma 4.2, and Propositions 5.1 and B.1, we see that it is enough to study the operator $(1 - \chi) e^{\Phi/h} \mathcal{U} \mathcal{M}^{-1} e^{-\Phi/h} (1 - \chi)$, where $\chi \in C_0^\infty(\mathbb{R})$ is such that $\chi(y_n) = 1$ on $\operatorname{Supp} \nabla \Phi$. But then, this operator coincides with $(1 - \chi) \mathcal{U} \mathcal{M}^{-1} (1 - \chi)$, and the result is obvious. \square

Now, if c satisfies (3.1), so does $c(x', x_n - ik)$, and thus, we deduce from the previous lemma that we have,

$$(5.23) \quad \|e^{\Phi/h} h \tilde{R}_k v\| \leq C h \|\langle h D_y \rangle e^{\Phi/h} v\|.$$

As a consequence, by (5.22), we deduce,

$$(5.24) \quad \begin{aligned} \langle e^{\Phi/h} \tilde{Q}v, e^{\Phi/h} \tilde{v} \rangle &= -\langle e^{\Phi/h} \tilde{Q}_1 v_1, e^{\Phi/h} v_1 \rangle + \langle e^{\Phi/h} Q_2 v_2, e^{\Phi/h} v_2 \rangle \\ &\quad + \mathcal{O}(h \|\langle h D_y \rangle e^{\Phi/h} v\|^2). \end{aligned}$$

We also compute,

$$-e^{\Phi/h} \tilde{Q}_1 e^{-\Phi/h} = -h^2 \Delta + \tilde{V}_1(y_n) - |\nabla \Phi|^2 + ik - ih(D_y \cdot \nabla \Phi + \nabla \Phi \cdot D_y),$$

and thus,

$$-\operatorname{Re} \langle e^{\Phi/h} \tilde{Q}_1 v_1, e^{\Phi/h} v_1 \rangle = \|h \nabla v_1\|^2 + \langle (\tilde{V}_1(y_n) - |\nabla \Phi|^2) e^{\Phi/h} v_1, e^{\Phi/h} v_1 \rangle.$$

Then, the result follows from (5.24) by applying Proposition 5.3 and by taking h small enough. \square

Remark 5.7. *It results from the proof that the dependence with respect to Φ of the constant C_2 is performed only through the estimates (5.11)-(5.12).*

As a consequence, if Φ depends on some extra-parameter and satisfies (5.11)-(5.12) uniformly with respect to this parameter, then the constant C_2 can be taken independent of it.

6. COMPARISON BETWEEN FORMAL AND TRUE SOLUTION

6.1. On $\{\phi < S(\mu)\}$. Now, we start by studying the region $\{\phi < S(\mu)\}$. The idea is to use the previous Agmon estimates in order to have informations on the decay of the first eigenfunction $\tilde{\mathbf{v}}$ of \tilde{Q} , and then to transfer these informations on \mathbf{v} by comparing $\tilde{\mathbf{v}}$ and \mathbf{v} .

Note that, by construction, we have $\tilde{V}_1 \geq \delta k^{2/3} \langle y_n \rangle$ everywhere (with some $\delta > 0$ constant). As a consequence, we see that the operator $\tilde{Q}_1 - z$ is invertible for any complex number z with $z = \mathcal{O}(h)$, and the norm of its inverse is $\mathcal{O}(k^{-2/3})$. Therefore, the 2×2 system of equations: $Qu = zu$ (with $u = (u_1, u_2)$), can be reduced to a scalar equation on u_2 , of the form,

$$(Q_2 + h^2 k^{-2/3} B(z))u_2 = zu_2,$$

with $\|B(z)\| = \mathcal{O}(1)$ uniformly. Since $h^2 k^{-2/3} = h^{4/3} |\ln h|^{-2/3} \ll h$, one can perform the standard regular-perturbation procedure, and conclude (also using Proposition C.1 in Appendix C) that there exists a unique discrete eigenvalue $\tilde{\rho}$ of \tilde{Q} near ρ , that coincides with ρ up to $\mathcal{O}(h^{4/3} |\ln h|^{-2/3})$. Finally, using, e.g., the WKB constructions of Section 4, we easily conclude that $\tilde{\rho}$ coincides with ρ up to $\mathcal{O}(h^\infty)$.

We have the following result:

Proposition 6.1. *For any multi-index α , the (conveniently normalized) eigenfunction $\tilde{\mathbf{v}}$ of \tilde{Q} , associated with the eigenvalue $\tilde{\rho}$, satisfies,*

$$(6.1) \quad \partial^\alpha (\tilde{\mathbf{v}} - \mathbf{w}) = \mathcal{O}(h^\infty e^{-\phi/h})$$

locally uniformly in $\{\phi < S(\mu)\}$. Moreover, for any $\varepsilon > 0$

$$(6.2) \quad \partial^\alpha \tilde{\mathbf{v}} = \mathcal{O}(e^{-(S(\mu)-\varepsilon)/h}),$$

locally uniformly on $\mathbb{R}^n \setminus \{\phi < S(\mu)\}$.

Proof. At first, we observe that, by arguments similar to those of [HeSj1] (that is, using the WKB solution \mathbf{w} constructed in Section 4, and comparing it with $\tilde{\Pi}\mathbf{w}$, where $\tilde{\Pi}$ is the spectral projector of \tilde{Q} onto $\text{Ker}(\tilde{Q} - \tilde{\rho})$), for

any multi-index α , one has,

$$(6.3) \quad \partial^\alpha(\tilde{\mathbf{v}} - \mathbf{w}) = \mathcal{O}(h^\infty),$$

uniformly in some (μ -dependent) neighborhood of 0.

Then, we consider the regularization $\tilde{\phi}$ of ϕ obtained by modifying it near Γ in the following way:

$$\tilde{\phi} = \tilde{\varphi}_2 + \chi_0(z/h)(\tilde{\varphi}_1 - \tilde{\varphi}_2)$$

where $z = z(y)$ is defined in (4.15), and $\chi_0 \in C^\infty(\mathbb{R}; [0, 1])$, $\chi_0(t) = 0$ for $t \leq -1$, $\chi_0(t) = 1$ for $t \geq 1$, and $0 \leq \chi' \leq 3/2$. In particular, $\tilde{\phi}$ is well defined on $\{\phi < S(\mu)\}$, and, by (4.15), we see that $\tilde{\phi}$ satisfies (5.11)-(5.12) locally uniformly, with $\lambda_h = 1$. Moreover, one also has,

$$\tilde{\phi} - \phi = \mathcal{O}(h^2) \quad ; \quad \nabla \tilde{\phi} - \nabla \phi = \mathcal{O}(h) \quad \text{a.e.},$$

and, for some $\delta = \delta(\mu) > 0$ small, $\partial^\alpha \tilde{\phi} = \mathcal{O}(1)$ on $\{\tilde{\phi} < \delta\}$.

Now, for $C > 0$ fixed large enough, we consider the function

$$(6.4) \quad \Phi_0 = \tilde{\phi} - Ch \ln \left(1 + \left(\frac{\tilde{\phi}}{Ch} - 1 \right) \chi(\tilde{\phi}/(Ch)) \right)$$

where $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = 0$ for $t \leq 1$, $\chi(t) = 1$ for $t \geq 2$, $0 \leq \chi' \leq 2$.

On the support of $\chi'(\tilde{\phi}/(Ch))$, we have $\tilde{\Phi} = \tilde{\varphi}_2$ e $\nabla \tilde{\phi} = \mathcal{O}(\sqrt{Ch})$. Therefore, on this set, we see that Φ_0 satisfies,

$$\partial^\alpha \Phi_0 = \mathcal{O}(1 + h^{1-|\alpha|/2}),$$

uniformly with respect to $C \geq 1$ and h small enough. On the other hand, away from $\text{Supp} \chi'(\tilde{\phi}/(Ch))$, we have either $\Phi_0 = \tilde{\phi}$, or $\Phi_0 = \tilde{\phi} - Ch \ln(\tilde{\phi}/Ch) = \tilde{\phi} - Ch \ln \tilde{\phi} + Ch \ln(Ch)$. In any case, we conclude that Φ_0 satisfies (5.11)-(5.12) with $\lambda_h = 1$, locally uniformly on $\{\phi < S(\mu)\}$, and uniformly with respect to the pair (C, h) such that $Ch \leq 1$.

We also have,

$$\tilde{p}_2(y, i\nabla \Phi_0(y)) = \tilde{p}_2(y, i(1 - F(\tilde{\phi}/(Ch)))\nabla \tilde{\phi}(y))$$

with $F(t) = ((t-1)\chi'(t) + \chi(t))/(1 + (t-1)\chi(t))$, and one can easily check that $|F(t)| \leq 4/t$. Therefore, $F(\tilde{\phi}/(Ch))\nabla \tilde{\phi}(y) \leq 4h|\nabla \tilde{\phi}|/\tilde{\phi}$, and thus, by a Taylor expansion, we find,

$$\begin{aligned} \tilde{p}_2(y, i\nabla \Phi_0(y)) &= \tilde{p}_2(y, i\nabla \tilde{\phi}(y)) - iF(\tilde{\phi}/(Ch))\nabla \tilde{\phi}(y)\nabla_\eta \tilde{p}_2(y, i\nabla \tilde{\phi}(y)) \\ &\quad + \mathcal{O}((Ch|\nabla \tilde{\phi}|/\tilde{\phi})^2). \end{aligned}$$

Since $\tilde{p}_2(y, i\nabla\tilde{\varphi}_2(y)) = 0$ and $\tilde{p}_2(y, i\nabla\tilde{\varphi}_1(y)) > 0$ on Γ_+ (see Appendix D), and $-i\nabla\tilde{\phi}(y)\nabla_\eta\tilde{p}_2(y, i\nabla\tilde{\phi}(y)) \geq \delta_0|\nabla\tilde{\phi}(y)|^2$ in some fixed neighborhood of 0 and for some uniform constant $\delta_0 > 0$, we conclude,

$$\tilde{p}_2(y, i\nabla\Phi_0(y)) \geq \delta_0 F(\tilde{\phi}/(Ch))|\nabla\tilde{\phi}(y)|^2 + \mathcal{O}((Ch|\nabla\tilde{\phi}|/\tilde{\phi})^2) + \mathcal{O}(h^2) + \mathcal{O}_\Gamma(h)$$

where the last $\mathcal{O}_\Gamma(h)$ is uniform with respect to C and supported near Γ . In particular, on $\{\tilde{\phi} \geq C^2h\}$, and if C is large enough,

$$\tilde{p}_2(y, i\nabla\Phi_0(y)) \geq C \frac{h}{2\tilde{\phi}} |\nabla\tilde{\phi}(y)|^2 + \mathcal{O}(h^2) + \mathcal{O}_\Gamma(h).$$

Since, near Γ , $|\nabla\tilde{\phi}|$ stays uniformly away from 0, we finally obtain, by taking C large enough,

$$(6.5) \quad \tilde{p}_2(y, i\nabla\Phi_0(y)) \geq C \frac{h}{4\tilde{\phi}} |\nabla\tilde{\phi}(y)|^2 \quad \text{on} \quad \{\tilde{\phi} \geq C^2h\},$$

uniformly for h small enough.

Now, for a given $\varepsilon > 0$ arbitrarily small, we set,

$$\Phi(y) = \chi_\varepsilon(\Phi_0(y)),$$

where $\chi_\varepsilon \in C^\infty(\mathbb{R}_+)$ is such that $\chi_\varepsilon(t) = t$ if $0 \leq t \leq S(\mu) - 3\varepsilon$, $\chi_\varepsilon(t) = S(\mu) - 2\varepsilon$ if $t \geq S(\mu) - \varepsilon$, $0 \leq \chi'_\varepsilon(t) \leq 1$ everywhere. Then, the function Φ is smooth on \mathbb{R}^n and it satisfies,

$$\begin{aligned} \Phi &= \Phi_0(y) \quad \text{on} \quad \{\phi \leq S(\mu) - 4\varepsilon\}; \\ \Phi &= S(\mu) - 2\varepsilon \quad \text{on} \quad \{\phi \geq S(\mu) - \frac{\varepsilon}{2}\}; \\ |\nabla\Phi| &\leq |\nabla\Phi_0| \quad \text{everywhere}; \\ \partial^\alpha\Phi &= \mathcal{O}(h^{-(2-|\alpha|)_+}) \quad \text{uniformly on } \mathbb{R}^n. \end{aligned}$$

Moreover, the last estimate is also uniform with respect to the pair (C, h) as long as $Ch \leq 1$.

Then, thanks to (6.5), and the fact that $|\nabla\phi|^2 \geq \delta\phi$ for some positive constant δ on $\{\phi \leq S(\mu) - \varepsilon\}$, we see that, (denoting by $\mathbf{1}_A$ the characteristic function of a set A),

$$\tilde{p}_2(y, i\nabla\Phi(y)) \geq C'h + \delta_3 \mathbf{1}_{\{\phi(y) \geq S(\mu) - \varepsilon/2\}}$$

on $\{\tilde{\phi} \geq C^2h\}$, with $\delta_3 > 0$ and $C' = C'(C) > 0$ arbitrarily large, depending on the choice of C (here, we also use the fact that $\tilde{p}_2(y, is\nabla\tilde{\varphi}_1(y)) > 0$ for $y \in \Gamma_+$ and $0 \leq s \leq 1$: See Appendix D). On the other hand, on the support

of $\nabla\Phi$, we have $|\nabla\Phi|^2 \leq |\nabla\Phi_0|^2 \leq (1 - Ch/\tilde{\phi})|\nabla\tilde{\phi}|^2$, and we also obtain on this set,

$$\tilde{V}_1(y) - (\nabla\Phi)^2 = \mu - y_n - (\nabla\Phi)^2 \geq C'h\langle y_n \rangle,$$

with $C' > 0$ arbitrarily large, while, away from $\text{Supp}\nabla\Phi$, we have $\tilde{V}_1(y) - (\nabla\Phi)^2 = \tilde{V}_1(y) \geq \delta k^{2/3}\langle y_n \rangle \gg h\langle y_n \rangle$.

Finally, on $\{\phi \leq C^2h\}$, one has $\Phi = \phi$, and thus $e^{\Phi/h} = \mathcal{O}(1)$.

Now, we apply the Agmon estimates of Proposition 5.5 with $v := \tilde{\mathbf{v}} - \mathbf{w}$ (and $\lambda_h = 1$). Then, thanks also to (4.32)-(4.34) and Remark 5.7, we obtain,

$$\begin{aligned} \frac{1}{C_2} \|h\nabla(e^{\Phi/h}v)\|^2 + (C' - C_2)h\|e^{\Phi/h}v_2\|_{\phi \geq C^2h}^2 + (C' - C_2)h\|\sqrt{\langle y_n \rangle}e^{\Phi/h}v_1\|^2 \\ = \mathcal{O}(\|v\|_{\phi \leq C^2h}) + \mathcal{O}(h^\infty), \end{aligned}$$

where the constant C_2 does not depend on the choice of C in (6.4), and where $C' = C'(C)$ tends to ∞ as $C \rightarrow +\infty$. Therefore, choosing C sufficiently large, and using (6.3), we conclude,

$$\|h\nabla(e^{\Phi/h}v)\| + \|\sqrt{\langle y_n \rangle}e^{\Phi/h}v_1\| + \|e^{\Phi/h}v_2\| = \mathcal{O}(h^\infty).$$

Finally, using repeatedly the partial differential equations satisfied by $\tilde{\mathbf{v}}$ and \mathbf{w} , and taking advantage of the classical ellipticity of \tilde{Q}_1 and Q_2 , together with the positivity of \tilde{V}_1 for $|y_n|$ large, we deduce in a standard way that, for all $s \in \mathbb{R}$,

$$(6.6) \quad \|\langle y_n \rangle^s e^{\Phi/h}v_1\|_{H^s(\mathbb{R}^n)} + \|e^{\Phi/h}v_2\|_{H^s(\mathbb{R}^n)} = \mathcal{O}(h^\infty),$$

and thus, for any $\alpha \in \mathbb{N}^n$, by standard Sobolev estimates,

$$\partial^\alpha v = \mathcal{O}(h^\infty e^{-\Phi/h}),$$

uniformly. Then, the results follows from the fact that, on $\{\phi \leq S(\mu) - 4\varepsilon\}$, one has $\Phi = \Phi_0 \geq \phi - Ch \ln(S(\mu)/Ch)$, while, on $\mathbb{R}^n \setminus \{\phi \leq S(\mu) - 4\varepsilon\}$, one has $\Phi \geq S(\mu) - 5\varepsilon$. \square

Now, we are able to compare the (complex-translated) resonant state \mathbf{v} with \mathbf{w} .

Proposition 6.2. *For any multi-index α , the first (conveniently normalized) eigenfunction \mathbf{v} of Q satisfies,*

$$(6.7) \quad \partial^\alpha(\mathbf{v} - \mathbf{w}) = \mathcal{O}(h^\infty e^{-\phi/h})$$

locally uniformly in $\{\phi < S(\mu)\}$. Moreover, for any $\varepsilon > 0$

$$(6.8) \quad \partial^\alpha \mathbf{v} = \mathcal{O}(e^{-(S(\mu)-\varepsilon)/h}),$$

locally uniformly on $\mathbb{R}^n \setminus \{\phi < S(\mu)\}$.

Proof. Let γ be the oriented circle centered at ρ with radius δh , where $\delta > 0$ is a small enough constant. Then, γ does not meet the spectrum of Q , and we can write the spectral projector Π of Q associated with ρ , as,

$$\Pi = \frac{1}{2i\pi} \oint_{\gamma} (z - Q)^{-1} dz.$$

We set,

$$\mathbf{v}_0 := \Pi \tilde{\mathbf{v}}.$$

Then, \mathbf{v}_0 is solution of $(Q - \rho)\mathbf{v}_0 = 0$ and is colinear with \mathbf{v} . Therefore, the result will be an easy consequence of Proposition 6.1 and the following lemma:

Lemma 6.3. *For any multi-index α and for any $\varepsilon > 0$, one has,*

$$(6.9) \quad \partial^\alpha (\mathbf{v}_0 - \tilde{\mathbf{v}}) = \mathcal{O}(e^{-(S(\mu) - \varepsilon)/h})$$

locally uniformly in \mathbb{R}^n .

Proof. Since $\tilde{Q}\tilde{\mathbf{v}} = \tilde{E}\tilde{\mathbf{v}}$, for any $z \in \gamma$, we have,

$$(6.10) \quad (Q - z)\tilde{\mathbf{v}} = (\tilde{E} - z)\tilde{\mathbf{v}} + r,$$

with $r := (Q - \tilde{Q})\tilde{\mathbf{v}} = ((\langle y_n \rangle + y_n - \mu)(1 - \chi_h(y_n - \mu))\tilde{\mathbf{v}}_1, 0)$ (where we have set $\tilde{\mathbf{v}} =: (\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2)$). In particular, by (6.6), for any $s \geq 0$ and any $\varepsilon > 0$, we see,

$$(6.11) \quad \|r\|_{H^s(\mathbb{R}^n)} = \mathcal{O}(e^{-(S(\mu) - \varepsilon)/h}).$$

Moreover, we deduce from (6.10),

$$(z - Q)^{-1}\tilde{\mathbf{v}} = (z - \tilde{E})^{-1}\tilde{\mathbf{v}} + (z - \tilde{E})^{-1}(z - Q)^{-1}r,$$

and thus,

$$(6.12) \quad \mathbf{v}_0 = \tilde{\mathbf{v}} + \frac{1}{2i\pi} \int_{\gamma} (z - \tilde{E})^{-1}(z - Q)^{-1}r dz.$$

Now, along γ , we have $(z - \tilde{E})^{-1} = \mathcal{O}(h^{-1})$, $(z - Q_1)^{-1} = \mathcal{O}(k^{-1})$, and $(z - Q_2)^{-1} = \mathcal{O}(h^{-1})$ (this last estimate comes from Appendix C and the fact that $Q_2 = \mathcal{U}(-h^2\Delta + V_2(x', x_n - ik))\mathcal{U}^{-1}$). Therefore,

$$\|hR_{2,1}(Q_1 - z)^{-1}hR_{1,2}(Q_2 - z)^{-1}\| = \mathcal{O}(hk^{-1}) = \mathcal{O}\left(\frac{1}{|\ln h|}\right).$$

In particular, for h small enough, the operator $I + hR_{2,1}(Q_1 - z)^{-1}hR_{1,2}(Q_2 - z)^{-1}$ is invertible with uniformly bounded inverse, and one easily deduces

that $\|(z - Q)^{-1}\| = \mathcal{O}(h^{-1})$. Since one also has $(z - \tilde{E})^{-1} = \mathcal{O}(h^{-1})$, we learn from (6.11)-(6.12),

$$\|\mathbf{v}_0 - \tilde{\mathbf{v}}\|_{L^2} = \mathcal{O}(e^{-(S(\mu)-\varepsilon)/h}).$$

The same estimate for the H^s -norms is proved in the same way, by applying any arbitrary power of Q to (6.12), and the result follows by Sobolev estimates. \square

Now, to complete the proof of the proposition, we just observe that, by construction, \mathbf{v} and \mathbf{v}_0 are co-linear, and $\|\mathbf{v}_0\| = 1 + \mathcal{O}(e^{-(S(\mu)-\varepsilon)/h})$. therefore, we can take $\mathbf{v} = \lambda \mathbf{v}_0$ with $\lambda = 1 + \mathcal{O}(e^{-(S(\mu)-\varepsilon)/h})$, and the result follows from Lemma 6.3 and Proposition 6.1. \square

6.2. Near $\{y_n = \mu\}$.

6.2.1. *A priori estimates.* We first prove the following a priori estimates:

Proposition 6.4. *For any $\alpha \in \mathbb{N}^n$, there exists $N_\alpha \geq 0$ such that, for any $N \geq 1$, the eigenfunction $\tilde{\mathbf{v}}$ satisfies,*

$$(6.13) \quad \partial^\alpha(\tilde{\mathbf{v}} - \mathbf{w}) = \mathcal{O}(h^{-N_\alpha} e^{-S(\mu)/h})$$

locally uniformly in $\{y_n \geq \mu - (Nk)^{2/3}\}$ (where $k := h \ln h^{-1}$).

Remark 6.5. *An important feature of the last estimate is that the number N_α does not depend on the choice of N .*

Proof. Again, the proof is based on Agmon estimates, but we have to be more precise on the construction of the weight function near $\{y_n = \mu\}$.

Fix $\varepsilon > 0$ small enough, and let $\chi \in C^\infty(\mathbb{R}; [0, 1])$ such that $\chi = 1$ on $[\mu - \varepsilon, +\infty)$, and $\chi = 0$ on $(-\infty, \mu - 2\varepsilon]$. We set,

$$(6.14) \quad \tilde{\phi}_0(y_n) = \chi(y_n) \left(S(\mu) - \frac{2}{3}(\mu - y_n)^{3/2} \right)$$

and, for $N \geq 1$ large enough and $y_n \in [0, \mu)$, we define,

$$(6.15) \quad \Phi_N(y_n) = \tilde{\chi} \left(\tilde{\phi}_0(y_n) + 3k(S(\mu) - \tilde{\phi}_0(y_n))^{1/3} + 3Nk \right)$$

where $\tilde{\chi} \in C^\infty(\mathbb{R}_+)$ depends on Nk and is such that $\tilde{\chi}(s) = s$ for $s \leq S(\mu)$, $\tilde{\chi}(s) = S(\mu) + Nk$ for $s \geq S(\mu) + 2Nk$, $0 \leq \tilde{\chi}' \leq 1$ everywhere, and, for all $\ell \geq 0$, $\tilde{\chi}^{(\ell)} = \mathcal{O}((Nk)^{-(\ell-1)+})$ uniformly.

Then, Φ_N is well defined and smooth on $(-\infty, \mu)$, and we have $\Phi_N = S(\mu) + Nk$ when $\tilde{\phi}_0 + 3k(S(\mu) - \tilde{\phi}_0)^{1/3} \geq S(\mu) - Nk$. The latter condition is

implied by $S(\mu) - \tilde{\phi}_0 \leq Nk$, and thus, if we extend $\tilde{\Phi}_N$ by the constant value $S(\mu) + Nk$ on $\{\tilde{\phi}_0 \geq S(\mu) - Nk\}$, we obtain a smooth function on \mathbb{R} , that satisfies,

$$\begin{aligned} \Phi'_N &= \tilde{\chi}' \left(\tilde{\phi}_0 + 3k(S(\mu) - \tilde{\phi}_0)^{1/3} + 3Nk \right) \left(1 - \frac{k}{(S(\mu) - \tilde{\phi}_0)^{2/3}} \right) \tilde{\phi}'_0 \\ (6.16) \quad &= \mathcal{O}(1). \end{aligned}$$

Moreover, one can check that, for $\ell \geq 1$, one has,

$$(6.17) \quad \tilde{\phi}_0^{(\ell)} = \mathcal{O} \left((S(\mu) - \tilde{\phi}_0)^{1-2\ell/3} \right) \text{ in } (\mu - 2\varepsilon, \mu).$$

In particular, $\tilde{\phi}'_0 = \mathcal{O}((S(\mu) - \tilde{\phi}_0)^{1/3})$, and, since $S(\mu) - \tilde{\phi}_0 \in [Nk, 3Nk]$ on the support of $\tilde{\chi}'' \left(\tilde{\phi}_0 + 3k(S(\mu) - \tilde{\phi}_0)^{1/3} + 3Nk \right)$, and $\tilde{\chi}^{(\ell)} = \mathcal{O}((Nk)^{-(\ell-1)+})$, we deduce that, for all $\ell \geq 2$,

$$\Phi_N^{(\ell)} = \mathcal{O}((Nk)^{1-2\ell/3})$$

uniformly in $(\mu - 2\varepsilon, \mu)$. Moreover, by construction, Φ_N is constant on both $(-\infty, \mu - 2\varepsilon]$ and $[\mu, +\infty)$, and since $S(\mu) - \tilde{\phi}_0$ vanishes at $y_n = \mu$ only, we also have $\partial^\alpha \Phi_N = \mathcal{O}(1)$ uniformly on $[\mu - 2\varepsilon, \mu - \varepsilon]$. Therefore, Φ_N satisfies (5.12) with $\lambda(h) = k^{-1/3}$ and $\Omega = \tilde{\Omega} := \{y \in \mathbb{R}^n; \mu - \varepsilon \leq y_n \leq \mu\}$, and we can apply Proposition 5.5 with $\Phi = \Phi_N$, $\lambda_h = k^{-1/3}$, and $v = \tilde{\mathbf{v}} - \mathbf{w}$. Then, taking into account (4.32)-(4.34) and the fact that $\tilde{\rho} - \rho = \mathcal{O}(h^\infty)$, for any $\delta > 0$, we obtain,

$$\begin{aligned} &\frac{1}{C_2} \|h \nabla (e^{\Phi_N/h} v)\|^2 + \frac{1}{C_2} \langle \tilde{p}_2(y, i \nabla \Phi_N(y)) e^{\Phi_N/h} v_2, e^{\Phi_N/h} v_2 \rangle \\ &\quad + \langle \left(\tilde{V}_1(y_n) - (\nabla \Phi_N)^2 \right) e^{\Phi_N/h} v_1, e^{\Phi_N/h} v_1 \rangle \\ &\leq C_2 h \|e^{\Phi_N/h} v_1\|^2 + C_2 h k^{-1/3} \|e^{\Phi_N/h} v_2\|_{\tilde{\Omega}_\delta}^2 + C_2 h \|e^{\Phi_N/h} v_2\|^2 \\ &\quad + \varepsilon(h) e^{2c_1/h} + C_2 h^{-2N_0} e^{2c_2/h} + C_2 h^{-2N_0} e^{2c_3/h}, \end{aligned}$$

where $C_2 > 0$ is a constant, the quantities N_0, φ are those appearing in (4.32)-(4.34), $\varepsilon(h) = \mathcal{O}(h^\infty)$, and where we have set,

$$\begin{aligned} c_1 &:= \sup_{\tilde{\mathcal{V}}_2} (\Phi_N - \varphi); \\ c_2 &:= \sup_{\tilde{\mathcal{V}}_2 \cap \{y_n \geq g(y')\}} (\Phi_N - S(\mu)); \\ c_3 &:= \sup_{\tilde{\mathcal{V}}_2 \setminus \tilde{\mathcal{V}}_1} (\Phi_N - \varphi). \end{aligned}$$

Now, using Lemma E.1, we see,

$$\Phi_N - \varphi \leq 3k(S(\mu) - \tilde{\phi}_0)^{1/3} + 3Nk \leq 3(N + S(\mu)^{1/3})k,$$

and thus, for h small enough,

$$e^{2c_1/h} \leq h^{-6(N+S(\mu)^{1/3})}.$$

Moreover, since $\Phi_N \leq S(\mu) + Nk$ everywhere, we have,

$$e^{2c_2/h} = h^{-2N}.$$

Finally, since $\varphi > S(\mu)$ on $\tilde{\mathcal{V}}_2 \setminus \tilde{\mathcal{V}}_1$, we have $c_3 < 0$ and thus $e^{2c_3/h} = \mathcal{O}(h^\infty)$.

Summing up, for N large enough, we have proved,

$$\begin{aligned} & \frac{1}{C_2} \|h \nabla(e^{\Phi_N/h} v)\|^2 + \frac{1}{C_2} \langle \tilde{p}_2(y, i \nabla \Phi_N(y)) e^{\Phi_N/h} v_2, e^{\Phi_N/h} v_2 \rangle \\ & \quad + \langle \left(\tilde{V}_1(y_n) - (\nabla \Phi_N)^2 \right) e^{\Phi_N/h} v_1, e^{\Phi_N/h} v_1 \rangle \\ (6.18) \quad & C_2 h \|e^{\Phi_N/h} v_1\|^2 + C_2 h k^{-1/3} \|e^{\Phi_N/h} v_2\|_{\tilde{\Omega}_\delta}^2 \\ & \quad + C_2 h \|e^{\Phi_N/h} v_2\|^2 + C_2 h^{-2(N+N_0)} + C_N, \end{aligned}$$

where $C_N > 0$ is an h -dependent constant. Now, for N sufficiently large, on $\text{Supp} \nabla \Phi_N$, we have $\tilde{V}_1(y_n) = \mu - y_n$. Therefore, by (6.16), on $\tilde{\Omega} \cap \text{Supp}(\nabla \Phi_N)$, we have,

$$\begin{aligned} \tilde{V}_1(y_n) - (\nabla \Phi_N)^2 & \geq \mu - y_n - (1 - k/(S - \tilde{\phi}_0)^{2/3})^2 (\nabla \tilde{\phi}_0)^2 \\ & \geq (\mu - y_n) \left(\frac{2k}{(S - \tilde{\phi}_0)^{2/3}} - \frac{k^2}{(S - \tilde{\phi}_0)^{4/3}} \right). \end{aligned}$$

Since, on this set, we also have $k(S - \tilde{\phi}_0)^{-2/3} \leq k^{1/3}$ and $S - \tilde{\phi}_0 = \mathcal{O}((\mu - y_n)^{3/2})$, we conclude that, for h small enough,

$$\tilde{V}_1(y_n) - (\nabla \Phi_N)^2 \geq \frac{k}{C_3} \quad \text{on } \tilde{\Omega} \cap \text{Supp}(\nabla \Phi_N),$$

where $C_3 > 0$ is a large enough constant. On the other hand, away from $\text{Supp}(\nabla \Phi_N)$, we have $\tilde{V}_1(y_n) - (\nabla \Phi_N)^2 = \tilde{V}_1(y_n) \geq k^{2/3} \langle y_n \rangle / C_3$, and thus, in any case,

$$(6.19) \quad \tilde{V}_1(y_n) - (\nabla \Phi_N)^2 \geq \frac{k \langle y_n \rangle}{C_3} \quad \text{on } \{y_n \geq \mu - \varepsilon\}.$$

Moreover, since $\nabla \tilde{\phi}_0 = 0$ on $\{y_n = \mu\}$, while $V_2 > 0$ there, one easily checks that, if ε and δ are chosen sufficiently small, we have,

$$(6.20) \quad \tilde{p}_2(y, i \nabla \Phi_N(y)) \geq c_\mu \quad \text{on } \{y_n \geq \mu - \varepsilon - \delta\},$$

where $c_\mu > 0$ is a constant.

On the other hand, on $\{y_n \leq \mu - \varepsilon\}$, using Lemma E.1, we see that,

$$\Phi_N \leq \text{Min}\{S(\mu) - \delta_\mu, \tilde{\phi} + 4Nk\},$$

for some $\delta_\mu > 0$ independent of h . Therefore, by using Proposition 6.1, we obtain,

$$(6.21) \quad \|h\nabla(e^{\Phi_N/h}v)\|_{\{y_n \leq \mu-\varepsilon\}} + \|e^{\Phi_N/h}v\|_{\{y_n \leq \mu-\varepsilon\}} = \mathcal{O}(1).$$

Going back to (6.18), and using (6.19)-(6.21) (plus the fact that $\tilde{V}_1 = |y_n| + \mu$ for $y_n \leq 0$), we finally obtain,

$$\begin{aligned} & \|h\nabla(e^{\Phi_N/h}v)\|^2 + \|e^{\Phi_N/h}v_2\|^2 + k\|\sqrt{\langle y_n \rangle}e^{\Phi_N/h}v_1\|^2 \\ & \leq Ch\|e^{\Phi_N/h}v_1\|^2 + Chk^{-1/3}\|e^{\Phi_N/h}v_2\|_{\Omega_\delta}^2 \\ & \quad + Ch\|e^{\Phi_N/h}v_2\|^2 + Ch^{-2(N+N_0)} + C, \end{aligned}$$

where $C > 0$ is a constant (that may depend on N and μ). Since $k \gg h$ and $hk^{-1/3} \rightarrow 0$ as $h \rightarrow 0_+$, we conclude,

$$(6.22) \quad \|h\nabla(e^{\Phi_N/h}v)\|^2 + \|e^{\Phi_N/h}v_2\|^2 + k\|\sqrt{\langle y_n \rangle}e^{\Phi_N/h}v_1\|^2 = \mathcal{O}(h^{-2(N+N_0)}),$$

uniformly for $h > 0$ small enough. As before, using iteratively the equation, this permits us to obtain,

$$(6.23) \quad \|e^{\Phi_N/h}v_2\|_{H^s} + \|\langle y_n \rangle^s e^{\Phi_N/h}v_1\|_{H^s} = \mathcal{O}(h^{-(N+N_s)}),$$

for any $s \in \mathbb{R}$, where $N_s \geq 0$ does not depend on N .

Now, by Lemma E.1, for any $M \geq 1$, on $[\mu - (Mk)^{2/3}, \mu]$ we have $\tilde{\phi}_0 \geq S(\mu) - \frac{2}{3}Mk$. Therefore, taking $M = N' := 3N/2$, we obtain $\tilde{\phi}_0 \geq S(\mu) - Nk$, and thus (still by construction),

$$\Phi_N = S(\mu) + Nk \quad \text{for } y_n \geq \mu - (N'k)^{2/3}.$$

In particular, $e^{\Phi_N/h} \geq h^{-N}e^{S(\mu)/h}$ there, and thus, we deduce from (6.23),

$$(6.24) \quad \begin{aligned} & \|v_2\|_{H^s(y_n \geq \mu - (N'k)^{2/3})} + \|\langle y_n \rangle^s v_1\|_{H^s(y_n \geq \mu - (N'k)^{2/3})} \\ & = \mathcal{O}(h^{-N_s}e^{-S(\mu)/h}), \end{aligned}$$

Since $N' \rightarrow +\infty$ as $N \rightarrow +\infty$, the result of Proposition 6.4 follows by standard Sobolev estimates. \square

Proposition 6.6. *For any multi-index α , there exists $N_\alpha \geq 0$, such that, for all $N \geq 1$, the first eigenfunction \mathbf{v} of Q satisfies,*

$$(6.25) \quad \partial^\alpha(\mathbf{v} - \mathbf{w}) = \mathcal{O}(h^{-N_\alpha}e^{-S(\mu)/h})$$

locally uniformly in $\{y_n \geq \mu - (Nk)^{2/3}\}$.

Proof. The proof is similar to that of Proposition 6.2.

With the same notations, we see that $(Q - z)\tilde{\mathbf{v}} = (\tilde{E} - z)\tilde{\mathbf{v}} + r$, with r supported in $\{y_n \geq \mu - k^{2/3}\}$. Then, using (6.24), we find that, for any $s \geq 0$, $\|r\|_{H^s} = \mathcal{O}(h^{-N_s}e^{-S(\mu)/h})$ for some $N_s \geq 0$, and we deduce that $\|\mathbf{v}_0 - \tilde{\mathbf{v}}\| = \mathcal{O}(h^{-N'_0}e^{-S(\mu)/h})$ for some $N'_0 \geq 0$. In particular, $\|\mathbf{v}_0\| = 1 + \mathcal{O}(h^{-N'_0}e^{-S(\mu)/h})$, thus $\mathbf{v} = \lambda\mathbf{v}_0$ with $\lambda = 1 + \mathcal{O}(h^{-N'_0}e^{-S(\mu)/h})$, and the result follows by applying Proposition 6.4. \square

6.2.2. *Propagation.* Now, we are able to prove,

Proposition 6.7. *For any multi-index α , there exists $\delta_\alpha > 0$, such that, for all $N \geq 1$, the first eigenfunction \mathbf{v} of Q satisfies,*

$$(6.26) \quad \partial^\alpha(\mathbf{v} - \mathbf{w}) = \mathcal{O}(h^{\delta_\alpha N}e^{-S(\mu)/h})$$

uniformly with respect to $h > 0$ small enough, and locally uniformly with respect to y in $\{\mu - (Nk)^{2/3} \leq y_n \leq \mu + (Nk)^{2/3}\}$.

Proof. We proceed in a way very similar to that of [FuLaMa] (but in a simpler context, here). At first, for $\nu > 0$, we introduce the Bargmann transform T_ν , defined by the formula,

$$T_\nu u(y, \eta; h) := \int_{\mathbb{R}^n} e^{i(y-z)\eta/h - (y'-z')^2/2h - \nu(y_n - z_n)^2/2h} u(z) dz.$$

We also set $q_1(y, \eta) := \eta^2 + \mu - y_n$ and, for $y' \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$,

$$\gamma(t, y') := \exp tH_{q_1}(y', \mu; 0) = (y', \mu + t^2; 0, -t).$$

By the same arguments as at the beginning of Section 6.1, we see that the estimate (6.3) remains valid in any complex neighborhood of 0 of the form: $\{y \in \mathbb{C}^n; |\operatorname{Re} y| \leq 1/C; |\operatorname{Im} y| \leq C\sqrt{h}\}$, with a large enough constant $C > 0$. Moreover, because of the quadratic behaviour of $\tilde{\varphi}_2(y)$ near 0, we also have $\|\mathbf{w}(y + i\delta)\|_{L^2_{loc}} = \mathcal{O}(1)$ for any $\delta = \mathcal{O}(\sqrt{h})$. As a consequence, the same is true for $\tilde{\mathbf{v}}$, and we see that all the previous arguments can be extended to $\operatorname{Im} y_n = -\delta(h)$, if $\delta(h)$ satisfies,

$$0 \leq \delta(h) \leq C\sqrt{h}.$$

In particular, we obtain,

$$(6.27) \quad \partial^\alpha \mathbf{v}(y) = \mathcal{O}(h^{-N_\alpha}e^{-S(\mu)/h}),$$

locally uniformly in $\{\operatorname{Re} y_n \geq \mu, -C\sqrt{h} \leq \operatorname{Im} y_n \leq 0\}$. (Observe that, since $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are real on the real, one has $\operatorname{Re} \tilde{\varphi}_j(y', y_n - i\delta(h)) = \tilde{\varphi}_j(y) +$

$\mathcal{O}(\delta(h)^2)$.) Therefore, taking $\nu = 1$ and making a complex change of contour in the expression of $T_1 \mathbf{v}$, it is easy to see that, if $(y, \eta) \in \mathbb{R}^{2n}$ is fixed (independently of μ), with $\eta_n > 0$ and $y_n > \mu$, then,

$$T_1 \mathbf{v} = \mathcal{O}(e^{-1/\sqrt{h}} e^{-S(\mu)/h}) = \mathcal{O}(h^\infty) e^{-S(\mu)/h},$$

uniformly near (y, η) . In particular, for any $t \leq -1$ and $y' \in \mathbb{R}^{n-1}$, we have,

$$\gamma(t, y') \notin FS(e^{S(\mu)/h} \mathbf{v}),$$

where $FS(e^{S(\mu)/h} \mathbf{v})$ stands for the frequency set of $e^{S(\mu)/h} \mathbf{v}$ (see, e.g., [GuSt, Ma6]). By the standard result of propagation of the frequency set, we deduce that $\gamma(t, y')$ stays away from $FS(e^{S(\mu)/h} \mathbf{v})$ as long as the *a priori* estimate (6.27) remains valid in some fix neighborhood of $\pi_y \gamma(t, y') := (y', \mu + t^2)$, that is, as long as t remains (strictly) negative. In other words, for any fixed $t < 0$ (arbitrarily close to 0), one has $\gamma(t, y') \notin FS(e^{S(\mu)/h} \mathbf{v})$.

On the other hand, if one has chosen the correct critical point in the definition of \mathbf{w} on $\{y_n \geq \mu\}$, one also has $\gamma(t, y') \notin FS(e^{S(\mu)/h} \mathbf{w})$ for any $t < 0$ close enough to 0. Therefore, $\gamma(t, y') \notin FS(e^{S(\mu)/h} (\mathbf{v} - \mathbf{w}))$, and thanks to Proposition 6.6, we can apply the same argument of propagation used in [FuLaMa] Section 6 (see also [GrMa]), and conclude to the existence of a constant $\delta > 0$, such that, for all $N \geq 1$,

$$(6.28) \quad T_1 \tilde{v}_N(\tilde{y}, \tilde{\eta}; \tilde{h}_N) = \mathcal{O}(e^{-\delta/\tilde{h}_N}) \quad \text{uniformly in } \mathcal{V}_N(\delta),$$

where we have set,

$$\tilde{v}_N(\tilde{y}) := e^{S(\mu)/h} (\mathbf{v} - \mathbf{w})(y'_0 + (Nk)^{\frac{1}{2}} \tilde{y}', \mu + (Nk)^{\frac{2}{3}} \tilde{y}_n)$$

$$\tilde{h}_N := (Nk)^{-1} h;$$

$$\mathcal{V}_N(\delta) := \{(\tilde{y}, \tilde{\eta}) \in \mathbb{R}^{2n}; |\tilde{y}| \leq \delta, (Nk)^{\frac{1}{6}} |\tilde{\eta}'| + |\tilde{\eta}_n| \leq \delta\}.$$

Here, $y'_0 \in \mathbb{R}^{n-1}$ is fixed arbitrarily, and the estimate is uniform with respect to $h > 0$ small enough and locally uniform with respect to y'_0 (see [FuLaMa], Proposition 6.8).

Note that, respect to [FuLaMa], here the situation is slightly simpler, because the asymptotic solution \mathbf{w} exists in a whole h -independent neighborhood of (y', μ) . This is the reason why the estimate (6.28) is valid for all N large enough, while that of [FuLaMa], Proposition 6.8 was valid for special values of N only.

As in [FuLaMa], the main point of the proof consists in observing that the function \tilde{v}_N is solution of an equation of the form,

$$(6.29) \quad \tilde{\mathbf{Q}}_N(\tilde{y}, \tilde{h}_N D_{\tilde{y}}; \tilde{h}_N) \tilde{v}_N = 0,$$

where the symbol $\tilde{\mathbf{q}}_N$ of $\tilde{\mathbf{Q}}_N$ satisfies,

$$\begin{aligned} \tilde{\mathbf{q}}_N(\tilde{y}, \tilde{\eta}; \tilde{h}_N) = & \left((Nk)^{1/3} |\tilde{\eta}'|^2 + \tilde{\eta}_n^2 \right) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \tilde{y}_n & 0 \\ 0 & \tilde{V}_N(\tilde{y}, \tilde{\eta}) \end{pmatrix} \\ & + \mathcal{O}(\tilde{h}_N (Nk)^{1/3}), \end{aligned}$$

with $\operatorname{Re} \tilde{V}_N > 0$ near $\mathcal{V}_N(\delta)$.

Using this equation, and in particular the fact that $\det \tilde{\mathbf{q}}_N(\tilde{y}, \tilde{\eta}; \tilde{h}_N) \neq 0$ in $\{|\tilde{y}| \leq \delta'; (Nk)^{1/3} |\tilde{\eta}'|^2 + \tilde{\eta}_n^2 \geq \delta\}$ if $\delta' > 0$ is small enough ($\delta > 0$ fixed), we deduce as in [FuLaMa] that, for all $m \geq 0$, one has,

$$(6.30) \quad \|\tilde{v}_N\|_{H^m(|\tilde{y}| \leq \frac{1}{2}\delta')} = \mathcal{O}(e^{-\delta_1/\tilde{h}_N}) = \mathcal{O}(h^{\delta_1 N}),$$

for some constant $\delta_1 > 0$. Then, the result follows from Sobolev estimates and from the fact that the previous estimate is locally uniform with respect to $y'_0 \in \mathbb{R}^{n-1}$. \square

Now, recall from Section 5.2 that the resonant state $\tilde{\mathbf{u}}$, solution of $\tilde{P}\tilde{\mathbf{u}} = \rho\tilde{\mathbf{u}}$, is related to \mathbf{v} by,

$$\tilde{\mathbf{u}}(y) = \mathbf{v}(y', y_n + ik).$$

We have,

Proposition 6.8. *For any $\varepsilon > 0$ small enough, set,*

$$\phi_\varepsilon := \operatorname{Max} \left(\tilde{\phi}_0, \min(\phi, S(\mu) - \varepsilon) \right),$$

where $\tilde{\phi}_0(y) := \chi_0(y_n) \left(S(\mu) - \frac{2}{3}(\mu - y_n)_+^{3/2} \right)$, with $\chi_0 \in C^\infty(\mathbb{R}; [0, 1])$, $\chi_0 = 1$ on $[\mu - \varepsilon_0 + \infty)$, $\chi_0 = 0$ on $(-\infty, \mu - 2\varepsilon_0]$, $\varepsilon_0 > 0$ fixed sufficiently small. Then, for any $\alpha \in \mathbb{N}^n$, there exists $N_\alpha \geq 0$, such that, for any $\varepsilon > 0$,

$$\partial^\alpha \tilde{\mathbf{u}}(y) = \mathcal{O}(h^{-N_\alpha} e^{-\phi_\varepsilon(y)/h}),$$

locally uniformly on \mathbb{R}^n .

Proof. We first observe that, by (6.6) and (6.23), the result is true for the function $\tilde{\mathbf{v}} - \mathbf{w}$ (instead of $\tilde{\mathbf{u}}$). Moreover, by construction and the proof of Lemma E.1, we see that it is also true for \mathbf{w} . Therefore, it is true for $\tilde{\mathbf{v}}$, and the proofs of Propositions 6.2 and 6.6 permit us to deduce that it is also true for \mathbf{v} .

Then, we use the elementary fact that,

$$(6.31) \quad (T_1 \tilde{\mathbf{u}})(y, \eta) = (T_1 \mathbf{v})(y', y_n + ik; \eta) = e^{k^2/2h + k\eta_n/h} (T_1 \mathbf{v})(y; \eta', \eta_n + k),$$

and, since $\|\mathbf{v}\|_{L^2} = 1$, we also know that $\|T_1 \mathbf{v}\|_{L^2}$ is uniformly bounded (actually, it is $\mathcal{O}(h^{\frac{3n}{4}})$). Therefore, $e^{-k\eta_n/h} T_1 \tilde{\mathbf{u}} \in L^2(\mathbb{R}^{2n})$, and,

$$(6.32) \quad \|e^{-k\eta_n/h} T_1 \tilde{\mathbf{u}}\|_{L^2} = \mathcal{O}(1),$$

uniformly as $h \rightarrow 0_+$. In particular, for any $\varepsilon > 0$, $e^{-\varepsilon k^{2/3} \langle \eta_n \rangle / h} T_1 \tilde{\mathbf{u}} \in L^2(\mathbb{R}^{2n})$.

Then, using the classical ellipticity (that is, as $|\eta| \rightarrow +\infty$) of the symbol of \tilde{P} , together with the analyticity of V_2 , one can show,

Lemma 6.9. *For any $\chi \in C_0^\infty(\mathbb{R}^n)$ and for any $M \geq 1$, there exists a constant $C > 0$ independent of μ such that,*

$$\sup_{|\eta| \geq C} |\langle \eta \rangle^M \chi(y) T_1 \tilde{\mathbf{u}}| = \mathcal{O}(e^{-1/Ch}),$$

uniformly for $h > 0$ small enough.

Proof. See Appendix F. □

Now, for y in some fixed arbitrary compact set, we write,

$$\begin{aligned} \tilde{\mathbf{u}}(y) &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{i(y-z)\eta/h - (y-z)^2/2h} \tilde{\mathbf{u}}(z) dz d\eta \\ &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} T_1 \tilde{\mathbf{u}}(y, \eta) d\eta, \end{aligned}$$

and thus, by Lemma 6.9 and (6.31),

$$\begin{aligned} \tilde{\mathbf{u}}(y) &= \frac{1}{(2\pi h)^n} \int_{|\eta| \leq C} e^{k^2/2h + k\eta_n/h} T_1 \mathbf{v}(y; \eta', \eta_n + k) d\eta + \mathcal{O}(e^{-1/Ch}) \\ &= \frac{1}{(2\pi h)^n} \int_{|\eta| \leq C} e^{k^2/2h + k\eta_n/h + i(y-z)\eta/h + ik(y_n - z_n)/h - (y-z)^2/2h} \mathbf{v}(z) dz d\eta \\ &\quad + \mathcal{O}(e^{-1/Ch}) \end{aligned} \tag{6.33}$$

In this integral, we perform the (singular) complex change of contour of integration given by,

$$(6.34) \quad \mathbb{R}^n \ni \eta \mapsto \eta + i\delta\chi(|\eta|) \frac{y-z}{|y-z|} \in \mathbb{C}^n,$$

with $\delta > 0$ small enough independent of μ , and where $\chi \in C_0^\infty([0, C]; [0, 1])$, $\chi = 1$ on $[0, C/2]$. We obtain,

$$\tilde{\mathbf{u}}(y) = \int_{|\eta| \leq C} \mathcal{O}(h^{-n} e^{k\eta_n/h - \delta\chi(|\eta|)(|y-z|/h - (y-z)^2/2h)}) \mathbf{v}(z) dz d\eta + \mathcal{O}(e^{-1/Ch}).$$

Then, taking advantage that we are in dimension $n \geq 2$, we use the polar coordinates in η : $\eta = r\omega$ ($r > 0$, $\omega \in S^{n-1}$), and since the integrated function is analytic with respect to ω , we can perform the (singular) complex change of contour of integration given by,

$$(6.35) \quad S^{n-1} \ni \omega \mapsto \omega + i\delta \frac{y-z}{|y-z|} \in \mathbb{C}^n.$$

This gives,

$$\tilde{\mathbf{u}}(y) = \int_{|\eta| \leq C} \mathcal{O}(h^{-n} e^{k\eta_n/h - \delta(\chi_1(|\eta|))|y-z|/h - (y-z)^2/2h}) \mathbf{v}(z) dz d\eta + \mathcal{O}(e^{-1/Ch}),$$

with $\chi_1(|\eta|) := |\eta| + \chi(|\eta|) \geq 1$, and thus,

$$(6.36) \quad \tilde{\mathbf{u}}(y) = \int \mathcal{O}(h^{-n-C} e^{-\delta|y-z|/h}) \mathbf{v}(z) dz + \mathcal{O}(e^{-1/Ch}).$$

Finally, we observe that, if ε is sufficiently enough, the Lipschitz function ϕ_ε is such that $\|\nabla \phi_\varepsilon\|_{L^\infty} = \mathcal{O}(\sqrt{\mu})$. Therefore, there exists a μ -independent constant $C > 0$ such that,

$$-\delta'|y-z| + \phi_\varepsilon(y) - \phi_\varepsilon(z) \leq -(\delta' - C\sqrt{\mu})|y-z|,$$

and thus, since $\delta' > 0$ can be taken independently of μ , we deduce from (6.36) and from the estimate on \mathbf{v} that, for μ sufficiently small, we have,

$$\tilde{\mathbf{u}} = \mathcal{O}(h^{-N_0} e^{-\phi_\varepsilon/h})$$

uniformly for $h > 0$ small enough, and with some constant $N_0 \geq 0$.

The estimates on the derivatives of $\tilde{\mathbf{u}}$ can be done in the same way. \square

Now, we denote by $\tilde{\mathbf{w}}$ the asymptotic solution constructed in Section 4 on the real domain. Then, we claim,

Proposition 6.10. *For any $\alpha \in \mathbb{N}^n$, there exists $\delta_\alpha > 0$, such that, for all $N \geq 1$, the resonant state $\tilde{\mathbf{u}}$ of \tilde{P} satisfies,*

$$(6.37) \quad \partial^\alpha(\tilde{\mathbf{u}} - \tilde{\mathbf{w}}) = \mathcal{O}(h^{\delta_\alpha N} e^{-S(\mu)/h})$$

uniformly with respect to $h > 0$ small enough, and locally uniformly with respect to y in $\{|y_n - \mu| \leq (Nk)^{2/3}\}$. Moreover, for any $\alpha \in \mathbb{N}^n$, there exists $N_\alpha \geq 0$ such that, for all $N \geq 1$, one has,

$$\partial^\alpha \tilde{\mathbf{u}} = \mathcal{O}(h^{-N_\alpha} e^{-S(\mu)/h}),$$

locally uniformly in $\{y_n \geq \mu\}$.

Proof. We use Proposition 6.6 and the representation of $\tilde{\mathbf{u}}$ given in (6.33). After the changes of contour (6.34)-(6.35), we obtain,

$$\tilde{\mathbf{u}}(y) = \frac{e^{k^2/2h}}{(2\pi h)^n} \int_{|\eta| \leq C} e^{i\theta/h + k\eta_n/h - \delta\chi_1(|\eta|)|y-z| - (y-z)^2/2h} \mathbf{v}(z) J(y, z, \eta) dz d\eta + \mathcal{O}(e^{-1/h}),$$

with $\theta := \delta k \chi_1(|\eta|) \frac{y_n - z_n}{|y-z|} + (y-z)\eta + k(y_n - z_n)$, and $J(y, z, \eta) = \mathcal{O}(1)$.

On $\{z_n \leq \mu - \varepsilon\}$ (with $\varepsilon > 0$ fixed small enough), the estimates on \mathbf{v} show that, for any $\varepsilon' > 0$, one has,

$$e^{-\delta|y-z|/h} \mathbf{v}(z) = \mathcal{O}(h^{-N_0} e^{-\delta|y-z|/h - \min(\phi, S - \varepsilon')/h}),$$

and thus, if in addition $y_n \geq \mu - (Nk)^{2/3}$ (and using the fact that $\|\nabla\phi\|_{L^\infty} = \mathcal{O}(\mu)$), for ε' small enough, on this set we obtain,

$$e^{-\delta|y-z|/h} \mathbf{v}(z) = \mathcal{O}(h^{-N_0} e^{-(S/h + \delta\varepsilon/2h - \varepsilon'/h)}) = \mathcal{O}(e^{-(S + \delta\varepsilon/4)/h}).$$

Moreover, on $\{\mu - \varepsilon \leq z_n \leq \mu - 2(Nk)^{2/3}\}$, we have,

$$e^{-\delta|y-z|/h} \mathbf{v}(z) = \mathcal{O}(h^{-N_0} e^{-\delta|y-z|/h - S/h + 2(\mu - z_n)^{3/2}/h}),$$

and thus, for $y_n \geq \mu - (Nk)^{2/3}$,

$$e^{-\delta|y-z|/h} \mathbf{v}(z) = \mathcal{O}(h^{-N_0} e^{-\delta(Nk)^{2/3}/2h - S/h}) = \mathcal{O}(h^\infty e^{-S/h}).$$

As a consequence, for $\tilde{\mathbf{u}}(y)$ with $y_n \geq \mu - (Nk)^{2/3}$, we obtain,

$$\begin{aligned} \tilde{\mathbf{u}}(y) &= \frac{e^{k^2/2h}}{(2\pi h)^n} \int_{\substack{|\eta| \leq C \\ z_n \geq \mu - 2(Nk)^{2/3}}} e^{i\theta/h + k\eta_n/h - \delta\chi_1(|\eta|)|y-z| - (y-z)^2/2h} \mathbf{v}(z) J(y, z, \eta) dz d\eta \\ &\quad + \mathcal{O}(h^\infty e^{-S/h}). \end{aligned}$$

In particular, for $y_n \geq \mu$, by Proposition 6.6 we deduce,

$$\tilde{\mathbf{u}}(y) = \mathcal{O}(h^{-N_0} e^{-S(\mu)/h}).$$

Moreover, if $|y_n - \mu| \leq (Nk)^{2/3}$, the previous arguments lead to,

$$\begin{aligned} \tilde{\mathbf{u}}(y) &= \frac{e^{k^2/2h}}{(2\pi h)^n} \int_{\substack{|\eta| \leq C \\ |z_n - \mu| \leq 2(Nk)^{2/3}}} e^{i\theta/h + k\eta_n/h - \delta\chi_1(|\eta|)|y-z| - (y-z)^2/2h} \mathbf{v}(z) J(y, z, \eta) dz d\eta \\ &\quad + \mathcal{O}(h^\infty e^{-S/h}). \end{aligned}$$

Then, using Proposition 6.7, we conclude,

$$\begin{aligned}
& \tilde{\mathbf{u}}(y) \\
&= \frac{e^{k^2/2h}}{(2\pi h)^n} \int_{\substack{|\eta| \leq C \\ |z_n - \mu| \leq 2(Nk)^{2/3}}} e^{i\theta/h + k\eta_n/h - \delta\chi_1(|\eta|)|y-z| - (y-z)^2/2h} \mathbf{w}(z) J(y, z, \eta) dz d\eta \\
& \quad + \mathcal{O}(h^{\delta N} e^{-S/h}),
\end{aligned}$$

for some $\delta > 0$ independent of N . Finally, taking y' close to some $y'_0 \in \mathbb{R}^{n-1}$, either \mathbf{w} is analytic around (y'_0, y_n) , or it is exponentially smaller than $e^{-S(\mu)/h}$. In the second case, we obtain $\tilde{\mathbf{u}}(y) = \mathcal{O}(h^{\delta N} e^{-S/h})$, while, in the first case, a change of contour of the type $z_n \mapsto z_n + ik\chi(|z_n - \mu|(Nk)^{-2/3})$ gives, by the same arguments as before, $\tilde{\mathbf{u}}(y) = \tilde{\mathbf{w}}(y) + \mathcal{O}(h^{\delta N} e^{-S/h})$. The same estimates holds for the derivatives, and this proves the proposition. \square

6.3. All around $\{\phi = S(\mu)\}$. We fix $\nu_0 > 0$ sufficiently small, but independent of μ , and, for $\nu \in (0, \nu_0/4]$, we consider the following spherical domain of \mathbb{R}^n ,

$$\mathcal{A}_\nu := \{\nu_0 - \nu < |y| < \nu_0 + \nu\}.$$

In this section, we plan to estimate $\tilde{\mathbf{u}}$ on \mathcal{A}_ν . By construction, $\tilde{\mathbf{w}} = 0$ in $\mathcal{A}_{2\nu}$ if $\mu \ll \nu_0$, and then we also have $\mathcal{A}_{2\nu} \subset \mathbb{R}^n \setminus \{\phi < S(\mu)\}$. Therefore, Propositions 6.8 and 6.10 already tell us that,

- $\partial^\alpha \tilde{u} = \mathcal{O}(h^{-N_\alpha} e^{-S(\mu)/h})$ on $\mathcal{A}_{2\nu} \cap \{y_n \geq \mu\}$;
- $\partial^\alpha \tilde{u} = \mathcal{O}(h^{\delta N} e^{-S(\mu)/h})$ on $\mathcal{A}_{2\nu} \cap \{|y_n - \mu| \leq 4(Nk)^{2/3}\}$;
- For any $\varepsilon > 0$, $\partial^\alpha \tilde{u} = \mathcal{O}(h^{-N_\alpha} e^{-\max(\tilde{\phi}_0, S(\mu) - \varepsilon)/h})$ on $\mathcal{A}_{2\nu} \cap \{y_n \leq \mu - (Nk)^{2/3}\}$,

where $N \geq 1$ is arbitrary, and the positive constants δ, N_α do not depend on N (note that the same estimates also hold for \mathbf{v}). Then, recording that $S(\mu) = \mathcal{O}(\mu^2)$, we consider a weight function $\psi \in C^\infty(\mathbb{R}^n; \mathbb{R}_+)$ such that,

- ψ is supported in $\mathcal{A}_{2\nu}$;
- $\psi \leq S(\mu)$ on $\{y_n \geq \mu + 4(Nk)^{2/3}\}$;
- $\psi \leq S(\mu) + \delta Nk$ on $\{y_n \leq \mu + 4(Nk)^{2/3}\}$;
- $\psi = S(\mu)$ on $\{y_n \geq \mu + 4(Nk)^{2/3}\} \cap \mathcal{A}_\nu$;
- $\psi = S(\mu) + \delta Nk$ on $\{y_n \leq \mu + 2(Nk)^{2/3}\} \cap \mathcal{A}_\nu$;
- $|\nabla \psi| \leq \mu$ on $\{y_n \leq \mu + 2(Nk)^{2/3}\} \cup \{y_n \geq \mu + 4(Nk)^{2/3}\}$;
- $\partial^\alpha \psi = \mathcal{O}(1)$ on $\{y_n \leq \mu + 2(Nk)^{2/3}\} \cup \{y_n \geq \mu + 4(Nk)^{2/3}\}$, ($|\alpha| \geq 2$);
- $\partial^\alpha \psi = \mathcal{O}((Nk)^{1-2|\alpha|/3})$ on $\{\mu + 2(Nk)^{2/3} \leq y_n \leq \mu + 4(Nk)^{2/3}\}$, ($|\alpha| \geq 1$).

Then, as before, the Agmon estimates applied with this weight function ψ lead to $\partial^\alpha \tilde{u} = \mathcal{O}(h^{-N_\alpha} e^{-\psi/h})$, and thus, in particular, for any $N \geq 1$, one has,

$$(6.38) \quad \partial^\alpha \tilde{u} = \mathcal{O}(h^{\delta N} e^{-S(\mu)/h})$$

uniformly on $\{y_n \leq \mu + 2(Nk)^{2/3}\} \cap \mathcal{A}_\nu$.

7. COMPLETION OF THE PROOF

We fix $\nu \in (0, \nu_0/4]$ small enough, independent of μ , and $\chi_\nu \in C^\infty(\mathbb{R}; [0, 1])$, such that $\chi_\nu = 1$ on $(-\infty, \nu_0 - \nu]$, $\chi_\nu = 0$ on $[\nu_0 + \nu, +\infty)$. We also fix $\chi_0 \in C^\infty(\mathbb{R}; [0, 1])$ such that $\chi_0 = 1$ on $(-\infty, 1]$, $\chi_0 = 0$ on $[2, +\infty)$. Then, for $y = (y', y_n) \in \mathbb{R}^n$ and $N \geq 1$, we set,

$$\chi_N(y) := \chi_\nu(|y|) \chi_0\left(\frac{y_n - \mu}{(Nk)^{2/3}}\right).$$

In particular, the support of $\nabla \chi_N$ is included in the set.

$$\begin{aligned} \mathcal{B}_N &:= \left(\mathcal{A}_\nu \cap \{y_n - \mu \leq 2(Nk)^{2/3}\} \right) \\ &\cup \left(\{(Nk)^{2/3} \leq y_n - \mu \leq 2(Nk)^{2/3}\} \cap \{|y| \leq \nu_0 + \nu\} \right). \end{aligned}$$

Then, we write,

$$(\operatorname{Im} \rho) \|\chi_N \tilde{\mathbf{u}}\|^2 = \operatorname{Im} \langle \chi_N \tilde{P} \tilde{\mathbf{u}}, \chi_N \tilde{\mathbf{u}} \rangle = -\operatorname{Im} \langle \chi_N [\tilde{P}, \chi_N] \tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle,$$

and, at this point, we can readily follow the arguments of [GrMa], Section 8 (with, in our case, $n_0 = 1$ and $n_\Gamma = 0$), and Theorem 3.1 follows.

APPENDIX

APPENDIX A. PROOF OF LEMMA 5.2

Setting $F(\theta, y) := \Psi(y + \frac{\theta}{2}, y - \frac{\theta}{2})$, we write,

$$(A.1) \quad a(y, \eta' + iF(\theta, y) + if_1) = a(y, \eta' + i\nabla \Phi(y) + if_1) + \tilde{a}(y, \eta', \theta),$$

where,

$$(A.2) \quad \tilde{a}(y, \eta', \theta) = i\theta \int_0^1 (\nabla_\theta F)(t\theta, y) \cdot (\nabla_{\eta'} a)(y, \eta' + iF(t\theta, y) + if_1) dt.$$

Now, by assumption (5.2), F satisfies,

$$|F| \leq \nu_0;$$

$$\partial^\alpha F = \mathcal{O} \left((1 + \lambda_h \mathbf{1}_{\substack{y \in \Omega_\delta \\ |\theta| < \delta}}) h^{1-|\alpha|} + \lambda_h h^{2-|\alpha|} \right) \quad (|\alpha| \geq 1),$$

where $\delta > 0$ is arbitrarily small.

In the following, for any set \mathcal{E} , we set,

$$\Lambda_h(\mathcal{E}) := 1 + \lambda_h \mathbf{1}_{\mathcal{E}},$$

where, as before, $\mathbf{1}_{\mathcal{E}}$ stands for the characteristic function of \mathcal{E} .

Then, we see on (A.2) that, for any $\alpha, \beta, \gamma \in \mathbb{N}^n$, we have,

$$\begin{aligned} \partial_\theta^\alpha \partial_y^\beta \partial_{\eta'}^\gamma \tilde{a}(y, \eta', \theta) &= \mathcal{O} \left(\langle \eta' \rangle^{2m} ((|\theta| + h) \Lambda_h(|\theta| < \delta; y \in \Omega_\delta) h^{-|\alpha| - |\beta|}) \right) \\ (A.3) \quad &+ \mathcal{O} \left(\langle \eta' \rangle^{2m} ((|\theta| + h) \lambda_h h^{1-|\alpha| - |\beta|}) \right). \end{aligned}$$

(Note that in the factor $(|\theta| + h)$, the term in h appears if $\alpha \neq 0$ only, and that the validity of the estimate extends to $(y, \eta') \in \mathbb{R}^n \times \mathbb{C}^n$, $|\operatorname{Im} \eta'|$ small enough.)

Now, using (A.1), we can split $b(y, \eta)$ into,

$$(A.4) \quad b(y, \eta) = b_0(y, \eta) + b_1(y, \eta) + b_2(y, \eta),$$

with,

$$\begin{aligned} b_0(y, \eta) &:= \frac{1}{(2\pi h)^n} \int e^{i(\eta' - \eta)\theta/h - g_1/h} a(y, \eta' + i\nabla\Phi(y) + if_1) d\eta' d\theta; \\ b_1(y, \eta) &:= \frac{1}{(2\pi h)^n} \int e^{i(\eta' - \eta)\theta/h - g_1/h} (\chi_0(\theta) - 1) a(y, \eta' + i\nabla\Phi(y) + if_1) d\eta' d\theta; \\ b_2(y, \eta) &:= \frac{1}{(2\pi h)^n} \int e^{i(\eta' - \eta)\theta/h - g_1/h} \chi_0(\theta) \tilde{a}(y, \eta', \theta) d\eta' d\theta. \end{aligned}$$

Making the change of contour of integration $\eta' \mapsto \eta' - if_1$ in the expression of b_0 , and using the fact that $\frac{1}{(2\pi h)^n} \int e^{i(\eta' - \eta)\theta/h} d\theta = \delta_{\eta' = \eta}$, we immediately obtain,

$$(A.5) \quad b_0(y, \eta) = a(y, \eta + i\nabla\Phi(y)).$$

On the other hand, if we set,

$$L := (1 + |\eta' - \eta|^2 + |\theta|^2)^{-1} (1 + (\eta' - \eta) \cdot hD_\theta + \theta \cdot hD_{\eta'}),$$

then, by integrations by parts, for any $N \geq 1$, we have,

$$(A.6) \quad b_1(y, \eta) = \frac{1}{(2\pi h)^n} \int e^{i(\eta' - \eta)\theta/h} c_N(y, \eta, \eta', \theta) d\eta' d\theta;$$

$$(A.7) \quad b_2(y, \eta) = \frac{1}{(2\pi h)^n} \int e^{i(\eta' - \eta)\theta/h} \tilde{c}_N(y, \eta, \eta', \theta) d\eta' d\theta,$$

with,

$$\begin{aligned} c_N(y, \eta, \eta', \theta) &:= ({}^t L)^N \left(e^{-g_1/h} (\chi_0(\theta) - 1) a(y, \eta' + i \nabla \Phi(y) + i f_1) \right); \\ \tilde{c}_N(y, \eta, \eta', \theta) &:= ({}^t L)^N \left(e^{-g_1/h} \chi_0(\theta) \tilde{a}(y, \eta', \theta) \right). \end{aligned}$$

Therefore, using (5.2), (A.3), and the fact that $g \geq 2\varepsilon_0$ on the support of $1 - \chi_0(y - y')$, we see that, for any $\alpha, \beta, \gamma \in \mathbb{N}^n$, we have,

$$\begin{aligned} \partial_y^\alpha \partial_\eta^\beta \partial_{\eta'}^\gamma c_N(y, \eta', \theta) &= \mathcal{O}(e^{-\varepsilon_0/h} (1 + |\eta' - \eta| + |\theta|)^{-N} \langle \eta' \rangle^{2m}; \\ \partial_y^\alpha \partial_\eta^\beta \partial_{\eta'}^\gamma \tilde{c}_N(y, \eta', \theta) &= \mathcal{O} \left((\Lambda_h(|\theta| < \delta; y \in \Omega_\delta) + h \lambda_h) (|\theta| + h) h^{-|\alpha|} \right) \\ &\quad \times (1 + |\eta' - \eta| + |\theta|)^{-N} \langle \eta' \rangle^{2m}, \end{aligned}$$

uniformly for $h > 0$ small enough, $(\theta, \eta) \in \mathbb{R}^{2n}$, and $(y, \eta') \in \mathbb{R}^n \times \mathbb{C}^n$, $|\operatorname{Im} \eta'|$ small enough.

In particular, by choosing N sufficiently large, we derive,

$$(A.8) \quad \partial_y^\alpha \partial_\eta^\beta b_1(y, \eta) = \mathcal{O}(e^{-\varepsilon_0/h} \langle \eta \rangle^{2m}).$$

Concerning b_2 , in (A.7) we make the (singular) change of contour of integration,

$$\mathbb{R}^n \ni \eta' \mapsto \eta' + i \delta \frac{\theta}{|\theta|},$$

where, as before, $\delta > 0$ is taken sufficiently small. We obtain,

$$\begin{aligned} \partial_y^\alpha \partial_\eta^\beta b_2(y, \eta) &= \frac{1}{(2\pi h)^n} \int \mathcal{O} \left(e^{-\delta|\theta|/h} (\Lambda_h(|\theta| < \delta; y \in \Omega_\delta) + h \lambda_h) (|\theta| + h) \right) \\ &\quad \times h^{-|\alpha|} \langle \theta \rangle^{-N} \langle \eta \rangle^{2m} d\theta \end{aligned}$$

and thus,

$$(A.9) \quad \partial_y^\alpha \partial_\eta^\beta b_2(y, \eta) = \mathcal{O}(\Lambda_h(y \in \Omega_\delta) h^{1-|\alpha|} \langle \eta \rangle^{2m}).$$

Then, Lemma 5.2 follows from (A.4), (A.5), (A.8), and (A.9).

APPENDIX B. AN EXOTIC PSEUDODIFFERENTIAL CALCULUS

In this appendix we have gathered some useful results concerning the pseudodifferential operators $A = \operatorname{Op}_h^W(a)$ with $a = a(x, \xi; h)$ smooth, such that a extends to a holomorphic function of ξ in a strip $\mathcal{A}_\nu := \{(x, \xi) \in \mathbb{R}^n \times \mathbb{C}^n; |\operatorname{Im} \xi| < \nu\}$ with $\nu > 0$, and, for all $\alpha, \beta \in \mathbb{N}^n$, it satisfies,

$$(B.1) \quad \partial_\xi^\beta a(x, \xi) = \mathcal{O}(\langle \operatorname{Re} \xi \rangle^m);$$

$$(B.2) \quad \partial_x^\alpha \partial_\xi^\beta a(x, \xi) = \mathcal{O}((1 + \lambda_h \mathbf{1}_\Omega(x)) h^{1-|\alpha|} \langle \operatorname{Re} \xi \rangle^m) \quad (|\alpha| \geq 1),$$

uniformly for $(x, \xi) \in \mathcal{A}_\nu$ and $h > 0$ small enough. Here, $m \in \mathbb{R}$ is some fix real number, Ω is some fix subset of \mathbb{R}^n , and $\lambda_h \geq 1$ is some given function of h such that $h\lambda_h = \mathcal{O}(1)$ uniformly.

We denote by $\mathcal{S}_m(\lambda_h, \Omega)$ the space symbols a such that the previous properties hold for some $\nu > 0$. For $t \in [0, 1]$, we also denote by $\text{Op}_h^t(a)$ the t -quantization of a (see, e.g., [Ma6]).

In order to describe the remainders terms that will appear in the symbolic calculus, we need to introduce another class of symbols. We say that a smooth symbol $a = a(x, \xi; h)$ is in $\tilde{\mathcal{S}}_m(\lambda_h, \Omega)$ if there exists some $\nu > 0$ such that a extends to a holomorphic function of ξ in \mathcal{A}_ν , and, for all $\alpha, \beta \in \mathbb{N}^n$, it satisfies,

$$(B.3) \quad \partial_x^\alpha \partial_\xi^\beta a(x, \xi) = \mathcal{O}((1 + \lambda_h \mathbf{1}_\Omega(x)) h^{-|\alpha|} \langle \text{Re } \xi \rangle^{2m}),$$

uniformly for $(x, \xi) \in \mathcal{A}_\nu$ and $h > 0$ small enough.

In particular, one has,

$$\mathcal{S}_m(\lambda_h, \Omega) \subset \tilde{\mathcal{S}}_m(\lambda_h, \Omega) \subset h^{-1} \mathcal{S}_m(\lambda_h, \Omega).$$

Proposition B.1. *With the previous notations, one has,*

- (i) *If $a \in \mathcal{S}_0(\lambda_h, \mathbb{R}^n)$, then $\text{Op}_h^W(a)$ is uniformly bounded on $L^2(\mathbb{R}^n)$.*
- (ii) *If $a \in \mathcal{S}_m(\lambda_h, \Omega)$, then, for any $t \in [0, 1]$ any $\delta > 0$, and any $u \in L^2$, one has,*

$$\| \langle hD_x \rangle^{-\frac{m}{2}} (\text{Op}_h^W(a) - \text{Op}_h^t(a)) \langle hD_x \rangle^{-\frac{m}{2}} u \| = \mathcal{O}(h\|u\| + h\lambda_h\|u\|_{L^2(\Omega_\delta)}),$$

where, as before, $\Omega_\delta := \{x \in \mathbb{R}^n; \text{dist}(x, \Omega) \leq \delta\}$, and where the estimate is uniform with respect to h and u .

- (iii) *Let $m, m' \in \mathbb{R}$, $a \in \mathcal{S}_m(\lambda_h, \Omega)$, and $b \in \mathcal{S}_{m'}(\lambda_h, \Omega)$. Then, for any $\delta > 0$, $a \#^W b \in \mathcal{S}_{m+m'}(\lambda_h, \Omega_\delta)$, and one has,*

$$a \#^W b = ab + hc,$$

with $c \in \tilde{\mathcal{S}}_{m+m'}(\lambda_h, \Omega_\delta)$.

- (iv) *If $a \in \tilde{\mathcal{S}}_0(\lambda_h, \Omega)$, then, for any $\delta > 0$, $\text{Op}_h^W(a)$ satisfies,*

$$\| \text{Op}_h^W(a) u \| = \mathcal{O}(\|u\| + \lambda_h\|u\|_{L^2(\Omega_\delta)}),$$

uniformly with respect to $u \in L^2(\mathbb{R}^n)$ and $h > 0$ small enough.

- (v) *Let $m, m' \in \mathbb{R}$, $a \in \tilde{\mathcal{S}}_m(\lambda_h, \Omega)$, and $b \in \mathcal{S}_{m'}(\lambda_h, \Omega)$. Then, for any $\delta > 0$, $a \#^W b \in \tilde{\mathcal{S}}_{m+m'}(\lambda_h, \Omega_\delta)$.*

- (vi) If $a \in \mathcal{S}_0(\lambda_h, \mathbb{R}^n)$ takes its values in \mathbb{R}_+ , then, there exists a constant $C > 0$ such that,

$$\mathrm{Op}_h^W(a) \geq -C\sqrt{h\lambda_h}.$$

Remark B.2. No complete symbolic calculus is available for this exotic class of symbols, but just up to $\mathcal{O}(h\lambda_h)$.

Remark B.3. It results from (iii) and (v) that, if $a_j \in \mathcal{S}_{m_j}(\lambda_h, \Omega)$ ($j = 1, 2, 3$), then, for any $\delta > 0$, the symbol $a_1 \#^W a_2 \#^W a_3$ can be written as,

$$(B.4) \quad a_1 \#^W a_2 \#^W a_3 = a_1 a_2 a_3 + hb,$$

with $c \in \tilde{\mathcal{S}}_{m_1+m_2+m_3}(\lambda_h, \Omega_\delta)$. Indeed, writing $a_1 \#^W a_2 = a_1 a_2 + hb_1$ with $b_1 \in \mathcal{S}_{m_1+m_2}(\lambda_h, \Omega_{\delta/2})$, it comes out,

$$a_1 \#^W a_2 \#^W a_3 = (a_1 a_2) \#^W a_3 + hb_1 \#^W a_3,$$

and since $a_1 a_2 \in \mathcal{S}_{m_1+m_2}(\lambda_h, \Omega)$ (see (B.5)-(B.6)), the result follows from a new application of (iii), and from (v). In particular, by (i) and (iv), this implies,

$$\mathrm{Op}_h^W(a_1) \mathrm{Op}_h^W(a_2) \mathrm{Op}_h^W(a_3) = \mathrm{Op}_h^W(a_1 a_2 a_3) + hB,$$

with,

$$\|\langle hD_x \rangle^{-\frac{m_1+m_2+m_3}{2}} B \langle hD_x \rangle^{-\frac{m_1+m_2+m_3}{2}} u\| = \mathcal{O}(\|u\| + \lambda_h \|u\|_{L^2(\Omega_\delta)}).$$

Proof. (i) The assumptions imply that $\partial_x^\alpha \partial_\xi^\beta a = \mathcal{O}(h^{-|\alpha|})$. Then, this is just an easy consequence of the Calderón-Vaillancourt Theorem: see, e.g., [Ma6], exercise 2.10.15.

(ii) It is well known that $\mathrm{Op}_h^W(a) = \mathrm{Op}_h^t(a_t)$, with (see, e.g., [Ma6], Theorem 2.7.1),

$$a_t(x, \xi) := \frac{1}{(2\pi h)^n} \int e^{i(\xi' - \xi)\theta/h} a(x + (t - \frac{1}{2})\theta, \xi') d\xi' d\theta,$$

and thus,

$$a_t(x, \xi) - a(x, \xi) := \frac{1}{(2\pi h)^n} \int e^{i(\xi' - \xi)\theta/h} (a(x + (t - \frac{1}{2})\theta, \xi') - a(x, \xi')) d\xi' d\theta.$$

Then, first making N integrations by parts by means of the operator $L := \frac{1}{1 + (\xi' - \xi)^2 + \theta^2} (1 + (\xi' - \xi) \cdot hD_\theta + \theta \cdot hD_{\xi'})$, and next performing the change of contour of integration given by,

$$\mathbb{R}^n \ni \xi' \mapsto \xi' + i\delta \frac{\theta}{|\theta|},$$

(with $\delta > 0$ small enough), we obtain as in Appendix A,

$$\partial_x^\alpha \partial_\xi^\beta (a_t(x, \xi) - a(x, \xi)) = \int \mathcal{O} \left(h^{-n} e^{-\delta|\theta|/h} \frac{(\Lambda_h(|\theta| \leq \delta; x \in \Omega_\delta) h^{-|\alpha|})}{(1 + (\xi' - \xi)^2 + \theta^2)^N} \right) \\ \times (|\theta| + h) \langle \xi' \rangle^m d\xi' d\theta,$$

with the notation $\Lambda_h(\mathcal{E}) := 1 + \lambda_h \mathbf{1}_{\mathcal{E}}$. Thus, if N has been taken sufficiently large, this implies,

$$\partial_x^\alpha \partial_\xi^\beta (a_t(x, \xi) - a(x, \xi)) = \mathcal{O}(\Lambda_h(x \in \Omega_\delta) h^{1-|\alpha|} \langle \xi \rangle^m).$$

Then, as in Appendix A, the result follows by taking a partition of unity adapted to the pair $(\Omega_\delta, \mathbb{R}^n \setminus \Omega_{\delta/2})$.

(iii) For any $X = (x, \xi) \in \mathbb{R}^{2n}$, one has,

$$a \#^W b(X) = \frac{1}{(2\pi h)^{4n}} \int_{\mathbb{R}^{8n}} e^{i[(X-Y)Y^* + (X-Z)Z^* + \sigma(Y^*, Z^*)]/h} a(Y) b(Z) \\ \times d(Y, Z, Y^*, Z^*),$$

where σ stands for the usual canonical symplectic form on \mathbb{R}^{2n} . Therefore, integrating first with respect to (y^*, z^*) (where we have set $Y^* = (y^*, \eta^*)$ and $Z^* = (z^*, \zeta^*)$), we obtain,

$$a \#^W b(X) = \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{4n}} e^{i[(\xi-\eta)\eta^* + (\xi-\zeta)\zeta^*]/h} a(x - \zeta^*, \eta) b(x + \eta^*, \zeta) \\ \times d(\eta, \eta^*, \zeta, \zeta^*).$$

Here, we observe that, by Leibniz formula, for $x \in \Omega$, we have,

$$(B.5) \quad \nabla_x(ab) = a \nabla_x b + b \nabla_x a = \mathcal{O}(\lambda_h \langle \xi \rangle^{m+m'}),$$

and, for $|\alpha| \geq 2$,

$$\begin{aligned} \partial_x^\alpha(ab) &= a \partial_x^\alpha b + b \partial_x^\alpha a + \sum_{\substack{\beta \leq \alpha \\ 1 < |\beta| < |\alpha|}} \mathcal{O}(|\partial^\beta a| \cdot |\partial^{\alpha-\beta} b|) \\ &= \mathcal{O}(\lambda_h h^{1-|\alpha|} \langle \text{Re } \xi \rangle^{m+m'} + \lambda_h^2 h^{2-|\alpha|} \langle \text{Re } \xi \rangle^{m+m'}) \\ (B.6) \quad &= \mathcal{O}(\lambda_h h^{1-|\alpha|} \langle \text{Re } \xi \rangle^{m+m'}), \end{aligned}$$

where the last estimate comes from the fact that $h\lambda_h = \mathcal{O}(1)$.

In particular $ab \in S_{m+m'}(\lambda_h, \Omega)$ and, making integrations by parts as before, and performing the change of contour,

$$\mathbb{R}^{2n} \ni (\eta, \zeta) \mapsto (\eta, \zeta) - i\delta' \left(\frac{\eta^*}{|\eta^*|}, \frac{\eta^*}{|\eta^*|} \right),$$

with $\delta' > 0$ small enough, we see that, for any $\delta > 0$, $a \#^W b$ belongs to $S_{m+m'}(\lambda_h, \Omega_\delta)$. Moreover, by an argument similar to that of the proof of

(ii), this also gives,

$$a\#^W b(X) = \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{4n}} e^{i[(\xi-\eta)\eta^* + (\xi-\zeta)\zeta^*]/h} a(x, \eta) b(x, \zeta) d(\eta, \eta^*, \zeta, \zeta^*) \\ + hc(X),$$

with,

$$\partial_x^\alpha \partial_\xi^\beta c(x, \xi) = \mathcal{O}(\Lambda_h(x \in \Omega_\delta) h^{-|\alpha|} \langle \operatorname{Re} \xi \rangle^{m+m'}).$$

Thus, $c \in \tilde{\mathcal{S}}_{m+m'}(\lambda_h, \Omega_\delta)$, and one finds,

$$a\#^W b(X) = a(X)b(X) + hc(X).$$

(iv) For any fix $\delta > 0$, by a partition of unity we can write $a = a_1 + a_2$ with $a_1 \in \tilde{\mathcal{S}}_0(1, \mathbb{R}^n)$, and $a_2 \in \tilde{\mathcal{S}}_0(\lambda_h, \Omega)$ supported in $\{x \in \Omega_{\delta/4}\}$. Then, the Calderón-Vaillancourt theorem tells us that $\operatorname{Op}_h^W(a_1)$ is uniformly bounded on $L^2(\mathbb{R}^n)$, and it just remains to study $\operatorname{Op}_h^W(a_2)$. Let $\chi \in C^\infty(\mathbb{R}^n; [0, 1])$ be such that $\operatorname{Supp} \chi \subset \Omega_\delta$ and $\chi = 1$ near $\Omega_{\delta/2}$. Then, we write,

$$\operatorname{Op}_h^W(a_2)u = \operatorname{Op}_h^W(a_2)(\chi u) + \operatorname{Op}_h^W(a_2)((1 - \chi)u),$$

and, still by the Calderón-Vaillancourt theorem, we have,

$$\|\operatorname{Op}_h^W(a_2)(\chi u)\| = \mathcal{O}(\lambda_h \|\chi u\|) = \mathcal{O}(\lambda_h \|u\|_{L^2(\Omega_\delta)}).$$

On the other hand, since the function $a_2((x + y)/2, \xi)(1 - \chi(y))$ vanishes identically in $\{|x - y| \leq \delta/4\}$, in the expression of $\operatorname{Op}_h^W(a_2)((1 - \chi)u)(x)$ we can make integrations by means of the operator,

$$L := |x - y|^{-2}(x - y) \cdot hD_\xi.$$

Then, for any $N \geq 1$, we obtain,

$$\operatorname{Op}_h^W(a_2)((1 - \chi)u) = \operatorname{Op}_h^W(b_N)u$$

where $b_N := (-1)^N L^N(a_2((x + y)/2, \xi)(1 - \chi(y)))$ satisfies,

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b_N = \mathcal{O}(h^N \lambda_h h^{-|\alpha| - |\beta|}).$$

Therefore, $\|\operatorname{Op}_h^W(b_N)\| = \mathcal{O}(\lambda_h h^N) = \mathcal{O}(h^{N-1})$, and the result follows by taking $N \geq 2$.

(v) For $a \in \tilde{\mathcal{S}}_m(\lambda_h, \Omega)$ and $b \in \mathcal{S}_{m'}(\lambda_h, \Omega)$, on Ω we have,

$$\nabla_x(ab) = a\nabla_x b + b\nabla_x a = \mathcal{O}((\lambda_h^2 + \lambda_h h^{-1}) \langle \operatorname{Re} \xi \rangle^{m+m'}) = \mathcal{O}(\lambda_h h^{-1} \langle \operatorname{Re} \xi \rangle^{m+m'}),$$

and, for $|\alpha| \geq 2$,

$$\begin{aligned}\partial_x^\alpha(ab) &= a\partial_x^\alpha b + b\partial_x^\alpha a + \sum_{\substack{\beta \leq \alpha \\ 1 < |\beta| < |\alpha|}} \mathcal{O}\left(|\partial^\beta a| \cdot |\partial^{\alpha-\beta} b|\right) \\ &= \mathcal{O}(\lambda_h^2 h^{1-|\alpha|} + \lambda_h h^{-|\alpha|} + \lambda_h^2 h^{1-|\alpha|}) \langle \operatorname{Re} \xi \rangle^{m+m'} \\ &= \mathcal{O}(\lambda_h h^{-|\alpha|}) \langle \operatorname{Re} \xi \rangle^{m+m'}.\end{aligned}$$

Therefore, ab is in $\widetilde{\mathcal{S}}_{m+m'}(\lambda_h, \Omega)$, and the rest of the proof proceeds exactly as in (iii).

(vi) We introduce the modified anti-Wick quantization, given by,

$$\widetilde{\operatorname{Op}}_h^{AW}(a) := \operatorname{Op}_h^W(\widetilde{a}),$$

where we have set,

$$\widetilde{a}(x, \xi) := \frac{1}{(\pi h)^n} \int a(y, \eta) e^{-\lambda_h^{1/2}(x-y)^2/h^{3/2} - \lambda_h^{-1/2}(\xi-\eta)^2/h^{1/2}} dy d\eta.$$

Then, by a second-order Taylor expansion of $a(y, \eta)$ at (x, ξ) , and using the fact that $\nabla_x^2 a = \mathcal{O}(\lambda_h h^{-1})$ and $\nabla_\xi^2 a = \mathcal{O}(1)$, we see that,

$$\widetilde{a} - a = \mathcal{O}(\sqrt{h\lambda_h}),$$

uniformly. In the same way, using that $\partial_x^\alpha \partial_\xi^\beta \widetilde{a} = \widetilde{\partial_x^\alpha \partial_\xi^\beta a}$, we obtain,

$$\partial_x^\alpha \partial_\xi^\beta (\widetilde{a} - a) = \mathcal{O}(\sqrt{h\lambda_h} h^{-|\alpha|}).$$

As a consequence, by the Calderón-Vaillancourt theorem,

$$(B.7) \quad \|\widetilde{\operatorname{Op}}_h^{AW}(a) - \operatorname{Op}_h^W(a)\| = \mathcal{O}(\sqrt{h\lambda_h}).$$

On the other hand, for any $u \in L^2(\mathbb{R}^n)$, a straightforward computation gives,

$$\langle \widetilde{\operatorname{Op}}_h^{AW}(a)u, u \rangle = \pi^{-n/2} h^{-7n/4} \lambda_h^{n/2} \int a(y, \eta) |H_u(y, \eta)|^2 e^{-\lambda_h^{1/2}y^2/h^{3/2}/h} dy d\eta,$$

where H_u is the (modified) Husimi function, defined by,

$$H_u(y, \eta) := \int e^{i[x\eta/h - \lambda_h^{1/2}x^2/(2h^{3/2}) - \lambda_h^{1/2}xy/h^{3/2}]} u(x) dx.$$

In particular, $\langle \widetilde{\operatorname{Op}}_h^{AW}(a)u, u \rangle \geq 0$, and the result follows by using (B.7). \square

APPENDIX C. AN ESTIMATE ON P_2

Here we prove,

Proposition C.1. *Let $E_1 < E_2$ be the first two eigenvalue of $P_2 = -h^2\Delta + V_2$, with V_2 satisfying (1.1), and let $k = h \ln(h^{-1})$. Then, for any $z \in \mathbb{C}$ such that $|z - E_1| = \delta h$ with $0 < \delta < \inf_{0 < h < 1} (E_2 - E_1)/h$, one has,*

$$\|(-h^2\Delta + V_2(x', x_n - ik) - z)^{-1}\| = \mathcal{O}(h^{-1}).$$

Proof. By a Taylor expansion, we have,

$$V(x', x_n - ik) - z = V_2(x) - ik\partial_{x_n} V_2(x) + k^2W,$$

with $W = \mathcal{O}(1)$, and thus,

(C.1)

$$-h^2\Delta + V_2(x', x_n - ik) - z = (I + (-ik\partial_{x_n} V_2(x) + k^2W)(P_2 - z)^{-1})(P_2 - z).$$

Moreover, the assumptions on V_2 imply that $(\partial_{x_n} V_2)^2 = \mathcal{O}(V_2)$. Therefore, for any $u \in L^2$, we have,

$$\begin{aligned} \|(\partial_{x_n} V_2)(P_2 - z)^{-1}u\|^2 &= \langle (\partial_{x_n} V_2)^2(P_2 - z)^{-1}u, (P_2 - z)^{-1}u \rangle \\ &\leq C \langle V_2(P_2 - z)^{-1}u, (P_2 - z)^{-1}u \rangle \\ &\leq C \langle P_2(P_2 - z)^{-1}u, (P_2 - z)^{-1}u \rangle \\ &\leq C \langle (1 + z(P_2 - z)^{-1})u, (P_2 - z)^{-1}u \rangle \\ &\leq C\|u\| \cdot \|(P_2 - z)^{-1}u\| + C|z| \cdot \|(P_2 - z)^{-1}u\|^2. \end{aligned}$$

Since $|z| = \mathcal{O}(h)$ and we also know that $\|(P_2 - z)^{-1}\| = \mathcal{O}(h^{-1})$, we deduce,

$$\|(\partial_{x_n} V_2)(P_2 - z)^{-1}u\|^2 = \mathcal{O}(h^{-1})\|u\|^2,$$

and thus,

$$\|(\partial_{x_n} V_2)(P_2 - z)^{-1}\| = \mathcal{O}(h^{-1/2}).$$

As a consequence,

$$\|(-ik\partial_{x_n} V_2(x) + k^2W)(P_2 - z)^{-1}\| = \mathcal{O}(kh^{-1/2} + k^2h^{-1}) = \mathcal{O}(\sqrt{h} \ln \frac{1}{h}),$$

and the result follows by taking the inverse in (C.1). \square

APPENDIX D. A GEOMETRIC ESTIMATE

Here we prove the following result, that we use in Section 5:

Lemma D.1. *The function $\tilde{\varphi}_1$ defined in Section 4 is such that, for any $s \in [0, 1]$ and for any $y \in \Gamma_+$,*

$$\tilde{p}_2(y, is\nabla\tilde{\varphi}_1(y)) > 0.$$

Proof. We first observe that

$$\frac{\partial}{\partial s} \tilde{p}_2(y, is\nabla \tilde{\varphi}_1(y)) = \partial_\eta \tilde{p}_2(y, is\nabla \tilde{\varphi}_1(y)) \cdot i\nabla \tilde{\varphi}_1(y) \leq -\frac{1}{C} s(\nabla \tilde{\varphi}_1(y))^2 \leq 0$$

and therefore, it is enough to prove the result for $s = 1$. Moreover, since we already know that $\tilde{p}_2(y, i\nabla \tilde{\varphi}_1(y)) \neq 0$ outside $\Pi_y \Gamma$, it is enough to prove it for $y \in \Gamma_+$ close enough to $\Pi_y \Gamma$, that is, for $y \in \Gamma_+$ such that

$$(y, i\nabla \tilde{\varphi}_1(y)) = \exp t H_{\tilde{p}_1}(\tilde{y}, i\nabla \tilde{\varphi}_2(\tilde{y}))$$

for some $t \in \mathbb{C} \setminus \{0\}$ small enough and $\tilde{y} \in \Pi_y \Gamma \cap \mathbb{R}^n$. In particular, we have

$$(y, i\nabla \tilde{\varphi}_1(y)) = (\tilde{y}, i\nabla \tilde{\varphi}_2(\tilde{y})) - t(2i\nabla \tilde{\varphi}_2(\tilde{y}), (0, \dots, 1)) + \mathcal{O}(|t|^2)$$

and thus, for y to be real it is necessary that $t = it'$ for some real t' . Moreover, in order that y is in Γ_+ , we need that $y_n > \tilde{y}_n$ (at least up to $\mathcal{O}(|t|^2)$), that is,

$$2t' \partial_n \tilde{\varphi}_2(\tilde{y}) > 0$$

and thus: $t' > 0$ (since $\partial_n \tilde{\varphi}_2(\tilde{y}) > 0$).

Now, since $\tilde{p}_2(\tilde{y}, i\nabla \tilde{\varphi}_2(\tilde{y})) = 0$, we obtain by a Taylor expansion,

$$\begin{aligned} \tilde{p}_2(y, i\nabla \tilde{\varphi}_1(y)) &= 2t' (\partial_y \tilde{p}_2)(\tilde{y}, i\nabla \tilde{\varphi}_2(\tilde{y})) \cdot \nabla \tilde{\varphi}_2(\tilde{y}) - it' (\partial_{\eta_n} \tilde{p}_2)(\tilde{y}, i\nabla \tilde{\varphi}_2(\tilde{y})) + \mathcal{O}(|t|^2) \\ &\geq \frac{t'}{C} (\tilde{y} \cdot \nabla \tilde{\varphi}_2(\tilde{y}) + \partial_n \tilde{\varphi}_2(\tilde{y})) > 0 \end{aligned}$$

This finishes the proof. \square

APPENDIX E. AN ESTIMATE ON $\tilde{\phi}_0$

We have,

Lemma E.1. *Let $\tilde{\phi}_0$ be the function defined in (6.14). Then, there exists $\varepsilon' > 0$ such that, for all $y \in \mathbb{R}^n$ with $y_n \leq \mu - \varepsilon$, one has,*

$$\tilde{\phi}_0(y_n) \leq \text{Min}\{\tilde{\phi}(y), S(\mu) - \varepsilon'\}.$$

Proof. By definition, for $\mu - 2\varepsilon \leq y_n < g(y')$ (where $g(y')$ is the function defining the caustic set of $\tilde{\phi}$), we have $(\partial_{y_n} \tilde{\phi}_0(y_n))^2 = |\nabla \tilde{\phi}(y)|^2 = \mu - y_n$, and $\partial_{y_n} \tilde{\phi}_0 \geq 0$. Therefore, $|\partial_{y_n} \tilde{\phi}(y)| \leq \partial_{y_n} \tilde{\phi}_0(y_n)$, and we also have $\tilde{\phi}(y', g(y')) \geq S(\mu) \geq \tilde{\phi}_0(g(y'))$. Therefore, integrating from $y_n = g(y') - t$ to $y_n = g(y')$ (with $0 \leq t \leq g(y') - \mu + 2\varepsilon$), we deduce,

$$\tilde{\phi}_0(y_n) \leq \tilde{\phi}(y) \text{ if } \mu - 2\varepsilon \leq y_n < g(y').$$

In particular, $\tilde{\phi}_0(y_n) \leq \tilde{\phi}(y)$ on $\{\tilde{\phi}(y) < S(\mu)\} \cap \{y_n \geq \mu - 2\varepsilon\}$. Since we also have $\sup_{y_n \leq \mu - \varepsilon} \tilde{\phi}_0(y_n) < S(\mu)$, and $\tilde{\phi}_0(y_n) = 0$ for $y_n \leq \mu - 2\varepsilon$, the result follows. \square

APPENDIX F. PROOF OF LEMMA 6.9

If $c = c(x)$ satisfies (3.1), we have,

$$\mathcal{U}\mathcal{M}^{-1} = \text{Op}_h^L(\tilde{c}),$$

with,

$$\begin{aligned} \tilde{c}(y, \eta) &= \frac{1}{(2\pi h)^n} \int e^{i(y-x)(\xi-\eta)/h} \times \\ &\quad \times c(x' - 2\xi_n(\xi' + \eta'), x_n - \frac{2}{3}(\eta_n^2 + \eta_n\xi_n + \xi_n^2) - 2|\eta'|^2) dx d\xi. \end{aligned}$$

Here, the oscillatory integral is well defined thanks to (3.1), and it defines a smooth function on \mathbb{R}^{2n} , holomorphic with respect to y in a complex strip around \mathbb{R}^n . Moreover, observing that,

$$|\eta| \cdot |\nabla \langle y_n \rangle^{1/2}| \leq |\eta| \langle y_n \rangle^{-1/2} = \mathcal{O}(\langle y_n - 2|\eta|^2 \rangle^{1/2}),$$

we see that, by (3.1), we can make a deformation of contour of integration in order to obtain,

$$e^{-4k\langle y_n \rangle^{1/2}/h} \mathcal{U}\mathcal{M}^{-1} e^{4k\langle y_n \rangle^{1/2}/h} = \text{Op}_h^L(\check{c}),$$

where \check{c} has properties similar to those of \tilde{c} . Then, by standard properties of T_1 (see, e.g., [Ma6]), we have,

$$T_1 e^{-4k\langle y_n \rangle^{1/2}/h} \mathcal{U}\mathcal{M}^{-1} e^{4k\langle y_n \rangle^{1/2}/h} = \text{Op}_h^L(\check{c}(y - \eta^*, y^*)) T_1,$$

where (y^*, η^*) stands for the dual variable of (y, η) . Next, using the fact that $\check{c}(y - \eta^*, y^*)$ is holomorphic with respect to y , for any $\varepsilon > 0$, we find,

$$\begin{aligned} e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 e^{-4k\langle y_n \rangle^{1/2}/h} \mathcal{U}\mathcal{M}^{-1} e^{4k\langle y_n \rangle^{1/2}/h} \\ = \text{Op}_h^L(c_\varepsilon(\eta, y - \eta^*, y^*)) e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1, \end{aligned}$$

where $c_\varepsilon(y, \eta, y^*, \eta^*)$ can be computed explicitly, and has essentially the same properties as $\check{c}(y - \eta^*, y^*)$, with estimates uniform with respect to $\varepsilon > 0$ sufficiently small.

Then, taking a cut-off function $\chi_1 = \chi_1(|\eta^*|)$ that is equal to 1 near 0, and using the fact that $T_1^*T_1 = 1$, we can write,

$$\begin{aligned} (F.1) \quad & \text{Op}_h^L(c_\varepsilon(\eta, y - \eta^*, y^*))e^{-\varepsilon k^{2/3}\langle \eta \rangle/h}T_1 \\ &= \text{Op}_h^L(\chi_1(|\eta^*|)(c_\varepsilon(\eta, y - \eta^*, y^*))e^{-\varepsilon k^{2/3}\langle \eta \rangle/h}T_1 \\ &\quad + \text{Op}_h^L((1 - \chi_1(|\eta^*|))(c_\varepsilon(\eta, y - \eta^*, y^*))K_\varepsilon e^{-\varepsilon k^{2/3}\langle \eta \rangle/h}T_1, \end{aligned}$$

with $K_\varepsilon := e^{-\varepsilon k^{2/3}\langle \eta \rangle/h}T_1^*T_1 e^{\varepsilon k^{2/3}\langle \eta \rangle/h}$.

A simple computation (see, e.g., [Ma6], Section 3) shows that the distributional kernel of K_ε is given by,

$$K_\varepsilon(y, \eta; z, \zeta) := (\pi h)^{\frac{n}{2}} e^{i(y-z)(\eta+\zeta)/2h - (y-z)^2/4h - (\eta-\zeta)^2/4h + \varepsilon k^{2/3}(\langle \zeta \rangle - \langle \eta \rangle)/h}.$$

Therefore, the distributional kernel of $\text{Op}_h^L((1 - \chi_1(|\eta^*|))(c_\varepsilon(\eta, y - \eta^*, y^*))K_\varepsilon$ is,

$$I_1(y, \eta; z, \zeta) := \frac{1}{2^n(\pi h)^{\frac{3n}{2}}} \int e^{\Phi_1/h} (1 - \chi_1(|\eta^*|)) c_\varepsilon(\eta, y - \eta^*, y^*) dy' d\eta' dy^* d\eta^*,$$

with,

$$\begin{aligned} \Phi_1 := & i(y - y')y^* + i(\eta - \eta')\eta^* + \frac{i}{2}(y' - z)(\eta' + \zeta) - \frac{1}{4}(y' - z)^2 \\ & - \frac{1}{4}(\eta' - \zeta)^2 + \varepsilon k^{2/3}(\langle \zeta \rangle - \langle \eta \rangle), \end{aligned}$$

while the distributional kernel of $\text{Op}_h^L(\chi_1(|\eta^*|)(c_\varepsilon(\eta, y - \eta^*, y^*)))$ is,

$$I_2(y, \eta; z, \zeta) := \frac{1}{(2\pi h)^n} \int e^{i(y-z)y^*/h + i(\eta-\zeta)\eta^*/h} \chi_1(|\eta^*|) c_\varepsilon(\eta, y - \eta^*, y^*) dy^* d\eta^*.$$

In I_1 , we can perform the change of contour of integration,

$$\eta' \mapsto \eta' - i\delta\eta^*/\langle \eta^* \rangle,$$

with $\delta > 0$ small enough and independent of μ . Since $k^{2/3}(\langle \zeta \rangle - \langle \eta \rangle) \leq k^{4/3} + (\eta - \zeta)^2$ and $k^{4/3}/h \ll 1$, we see that the resulting phase factor can be estimated by,

$$e^{-\delta|\eta^*|^2/2h\langle \eta^* \rangle - (y-z)^2/8h - (\eta-\zeta)^2/8h}.$$

As a consequence, since η^* stays away from 0 on the support of $1 - \chi_1(|\eta^*|)$, and since, for any $N \geq 1$ one has $|y^*|^{2N} \langle \eta_n^* + |y^*|^2 \rangle^{-N} e^{-\delta\langle \eta^* \rangle/4h} = \mathcal{O}(1)$, by the Calderón-Vaillancourt theorem we conclude that, for any $\chi \in C_0^\infty(\mathbb{R}^n)$, we have,

$$\begin{aligned} & \|\chi(y) \text{Op}_h^L((1 - \chi_1(|\eta^*|))(c_\varepsilon(\eta, y - \eta^*, y^*))K_\varepsilon e^{-\varepsilon k^{2/3}\langle \eta \rangle/h}T_1 \check{\mathbf{u}}\| \\ &= \mathcal{O}(e^{-\delta_1/h}) \|e^{-\varepsilon k^{2/3}\langle \eta \rangle/h}T_1 \check{\mathbf{u}}\|, \end{aligned}$$

where we have set,

$$\check{\mathbf{u}}(y) := e^{-4k\langle y_n \rangle^{1/2}/h} \tilde{\mathbf{u}}(y),$$

and where $\delta_1 > 0$ is independent of μ . In particular, if $\psi \in C_b^\infty(\mathbb{R}^{2n}; \mathbb{R}_+)$ is such that $\sup |\psi| \leq \delta_1/2$, we obtain,

$$\begin{aligned} & \|e^{\psi/h} \chi(y) \text{Op}_h^L((1 - \chi_1(|\eta^*|))(c_\varepsilon(\eta, y - \eta^*, y^*)) K_\varepsilon e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}})\| \\ (F.2) \quad & = \mathcal{O}(e^{-\delta_1/2h}) \|e^{\psi/h - \varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}}\|, \end{aligned}$$

On the other hand, for y in some arbitrarily large fixed compact set K , in I_2 we can successively perform the two changes of contour,

$$\begin{aligned} & \mathbb{R}^n \ni \eta^* \mapsto \eta^* + i\delta\chi_2(|\eta^*|)(\eta - \zeta)/|\eta - \zeta| \\ (F.3) \quad & S^{n-1} \ni \omega^* := \eta^*/|\eta^*| \mapsto \omega^* + i\delta(\eta - \zeta)/|\eta - \zeta|, \end{aligned}$$

where $\chi_2 \in C_0^\infty$ is supported inside $\{\chi_1 = 1\}$, $\chi_2 = 1$ near 0, and $\delta > 0$ depends on K only. This makes appear the exponential factor,

$$e^{-\delta(|\eta^*| + \chi_2(|\eta^*|))|\eta - \zeta|/h}.$$

Moreover, since $|\eta^*|$ remains bounded on the support of $\chi_1(|\eta^*|)$, if $\psi \in C_b^\infty(\mathbb{R}^{2n}; \mathbb{R}_+)$ is such that $\sup |\nabla \psi| \ll \delta$, writing: $\psi(y, \eta) = \psi(z, \zeta) + (y - z)\phi_1 + (\eta - \zeta)\phi_2$, then we see that we can also perform the change of contour,

$$\mathbb{R}^n \ni y^* \mapsto y^* + i\phi_1(y, z, \eta, \zeta),$$

and we easily conclude that,

$$\begin{aligned} & \|e^{\psi/h} \chi(y) \chi_1(|\eta^*|) \text{Op}_h^L(c_\varepsilon(\eta, y - \eta^*, y^*)) e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}}\| \\ & = \mathcal{O}\left(\|e^{\psi/h - \varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}}\|\right). \end{aligned}$$

Here, $\chi \in C_0^\infty(\mathbb{R}^n)$ is an arbitrary cut-off function supported in K .

Summing up, we have proved that, for any $\chi \in C_0^\infty(\mathbb{R}^n)$ and for any $\psi \in C_b^\infty(\mathbb{R}^{2n}; \mathbb{R}_+)$ with $\sup |\psi| + \sup |\nabla \psi|$ small enough (independently of μ), there exists a constant $C > 0$ (independent of μ) such that, for any $\varepsilon > 0$, we have,

$$(F.4) \quad \|\chi(y) e^{\psi/h - \varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \mathcal{U} \mathcal{U}^{-1} \check{\mathbf{u}}\| \leq C \|e^{\psi/h - \varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}}\|.$$

It is important to observe that, in all these computations, the estimates are uniform with respect to $\varepsilon > 0$. Actually, the presence of ε is required only to insure that the norms are finite.

Remark F.1. One can extract from the previous proof the following (maybe already well-know) result: If $a(x, \xi)$ is holomorphic with respect to ξ is a

complex strip Γ around \mathbb{R}^n ($n \geq 2$), and is bounded together with all its derivatives in $\mathbb{R}^n \times \Gamma$, then, for any cut-off function $\chi \in C_0^\infty(\mathbb{R}_+)$ such that $\chi = 1$ near 0, and for any $\varphi \in C_b^\infty(\mathbb{R}^n)$ with $\sup(|\varphi| + |\nabla \varphi|)$ small enough, the operator $e^{\varphi/h} \text{Op}_h^W(\chi(|\xi|)a(x, \xi))e^{-\varphi/h}$ is uniformly bounded on $L^2(\mathbb{R}^n)$. (Indeed, this can be seen by first making integrations by parts with respect to ξ , and then by performing changes of contour similar to those of (F.3).)

In the sequel, we also need,

Lemma F.2. *There exists a constant $C > 0$ such that, for all $\varepsilon, h > 0$ small enough, one has,*

$$\begin{aligned} & \|e^{-\varepsilon k^{2/3}\langle \eta \rangle/h} T_1 e^{-4k\langle y_n \rangle^{1/2}/h} \mathcal{U} \mathcal{A}^{-1} e^{4k\langle y_n \rangle^{1/2}/h} \check{\mathbf{u}}\| \\ & \leq C \|(1 + \frac{\langle \eta \rangle^{2C}}{\langle y_n - 2\eta^2 \rangle^C}) e^{-\varepsilon k^{2/3}\langle \eta \rangle/h} T_1 \check{\mathbf{u}}\|. \end{aligned}$$

Proof. Using (3.1), one can check that the symbol c_ε satisfies,

$$\partial_{y, \eta, \eta^*}^\alpha \partial_{y^*}^\beta c_\varepsilon = \mathcal{O}(1 + \langle y^* \rangle^{|\beta|} \langle y_n - \eta_n^* - 2|y^*|^2 \rangle^{-|\beta|/2}),$$

uniformly with respect to ε . Therefore, we learn from the Calderón-Vaillancourt Theorem the existence of some constant $C > 0$ such that,

$$\begin{aligned} & \|\text{Op}_h^L(c_\varepsilon(\eta, y - \eta^*, y^*)) e^{-\varepsilon k^{2/3}\langle \eta \rangle/h} T_1 \check{\mathbf{u}}\| \\ & \leq C \|(1 + \langle hD_y \rangle^{2C} \langle y_n - hD_{\eta_n} - 2(hD_y)^2 \rangle^{-C}) e^{-\varepsilon k^{2/3}\langle \eta \rangle/h} T_1 \check{\mathbf{u}}\|. \end{aligned}$$

Then, the result follows from standard properties of the transform T_1 , that permit us to replace hD_y by η , and hD_η by 0, in the previous estimate (see, e.g., [Ma6]). \square

Now, we set,

$$\check{P} := e^{-4k\langle y_n \rangle^{1/2}/h} \tilde{P} e^{4k\langle y_n \rangle^{1/2}/h},$$

and we observe that, for any integer $N \geq 1$, the operator,

$$(\check{P} - \rho)^N = e^{-4k\langle y_n \rangle^{1/2}/h} \mathcal{U} (P - \rho)^N \mathcal{U}^{-1} e^{4k\langle y_n \rangle^{1/2}/h},$$

is a sum of terms of the type: $a(y, hD_y) e^{-4k\langle y_n \rangle^{1/2}/h} \mathcal{U} \mathcal{A}^{-1} e^{4k\langle y_n \rangle^{1/2}/h}$, where $a(y, hD_y)$ is a 2×2 matrix of partial differential operator with polynomial coefficients, and $c = c(x)$ satisfies (3.1). Moreover, the leading term (that is, the one corresponding to the operator $a(y, hD_y)$ with higher degree in hD_y) is just $(-h^2 \Delta)^N \begin{pmatrix} (-1)^N & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, combining (F.4) with standard microlocal exponential weighted estimates (see [Ma6], Section 3.5), we see that the equation $(\check{P} - \rho)^N \check{\mathbf{u}} = 0$ implies that, for any $\chi \in C_0^\infty(\mathbb{R}^n)$, for any

$\psi \in C_b^\infty(\mathbb{R}^{2n}; \mathbb{R}_+)$ with $\sup |\psi| + \sup |\nabla \psi|$ small enough, and for any $\varepsilon > 0$ small enough, one has,

$$(F.5) \quad \| |\eta|^{2N} \chi(y) e^{\psi/h - \varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}} \| \leq C \| \langle \eta \rangle^{2N-1} e^{\psi/h - \varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}} \|,$$

where the constant C does not depend on ε , and depends on ψ through the quantities $\sup |\partial^\alpha \psi|$ only. In particular, we can take ψ such that $\psi = 0$ on $\{\chi(y) \neq 1\} \cup \{|\eta|^{2N} \leq 2C \langle \eta \rangle^{2N-1}\}$, and $\psi \geq \delta_0$ on $K \times \{|\eta| \geq C_0\}$ with $\delta_0, C_0 > 0$ independent of ε and μ , and where K is an arbitrary compact subset of the interior of $\{\chi = 1\}$. In this case, we deduce from (F.5),

$$(F.6) \quad \| |\eta|^{2N} e^{\psi/h - \varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}} \|_{\text{Supp} \psi} \leq 2C \| \langle \eta \rangle^{2N-1} e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}} \|_{\mathbb{R}^{2n} \setminus \text{Supp} \psi}$$

On the other hand, using Lemma F.2 and the equation $(\check{P} - \rho)^N \check{\mathbf{u}} = 0$, we also deduce,

$$\begin{aligned} & \| |y_n - \eta^2|^N e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}}_1 \| + \| |\eta|^{2N} e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}}_2 \| \\ & \leq C \| (\langle \eta \rangle^{2N-1} + \langle y_n - \eta^2 \rangle^{N-1}) (1 + \frac{\langle \eta \rangle^{2C}}{\langle y_n - 2\eta^2 \rangle^C}) e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}} \|, \end{aligned}$$

with some other constant $C > 0$, still independent of ε . Now, on $\{\eta^2 \geq 2|y_n|\}$, we have $\eta^2 - y_n \geq \frac{1}{2}\eta^2$, while $\langle \eta \rangle^{2C} \langle y_n - 2\eta^2 \rangle^{-C}$ is bounded. Therefore, we obtain,

$$\begin{aligned} & \| |\eta|^{2N} e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}} \|_{\eta^2 \geq 2|y_n|} \\ & \leq C \| (\langle \eta \rangle^{2N-1} + \langle y_n - \eta^2 \rangle^{N-1}) (1 + \frac{\langle \eta \rangle^{2C}}{\langle y_n - 2\eta^2 \rangle^C}) e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}} \|_{\eta^2 \leq 2|y_n|} \\ & \quad + C \| |\eta|^{2N-1} e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}} \|_{\eta^2 \geq 2|y_n|}, \end{aligned}$$

and thus, if $C_1 > 0$ is sufficiently large (in order to have $|\eta|^{2N} \geq \langle \eta \rangle^{2N-1}$ when $|\eta|^2 \geq C_1$),

$$(F.7) \quad \begin{aligned} & \| |\eta|^{2N} e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}} \|_{\eta^2 \geq \max(C_1, 2|y_n|)} \\ & \leq C_N \| \langle y_n \rangle^{N+C} e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}} \|_{\eta^2 \leq 2|y_n|} + C_N \| e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}} \|_{\eta^2 \leq C_1}, \end{aligned}$$

with $C_N > 0$ independent of ε . Setting $\theta_1 = \theta_1(y_n; h) := 4k \langle y_n \rangle^{1/2} / h$ and $\theta_2 = \theta_2(\eta; h) := k \langle \eta \rangle / h$, for $\eta^2 \leq 2|y_n|$, we write,

$$\begin{aligned} \langle y_n \rangle^{N+C} T_1 \check{\mathbf{u}} &= \langle y_n \rangle^{N+C} T_1 e^{-\theta_1} \tilde{\mathbf{u}} \\ &= \langle y_n \rangle^{N+C} e^{\theta_2 - \theta_1} (e^{\theta_1 - \theta_2} T_1 e^{-\theta_1} T_1^* e^{\theta_2}) e^{-\theta_2} T_1 \tilde{\mathbf{u}}, \end{aligned}$$

and since $\theta_2(\eta) - \theta_1(y_n) \leq -\frac{1}{2}\theta_1(y_n)$, and $\langle y_n \rangle^{N+C} e^{-\frac{1}{2}\theta_1(y_n)} = \mathcal{O}(1)$, we obtain,

(F.8)

$$\|\langle y_n \rangle^{N+C} e^{-\varepsilon k^{2/3}\langle \eta \rangle/h} T_1 \check{\mathbf{u}}\|_{\eta^2 \leq 2|y_n|} \leq C'_N \|(e^{\theta_1-\theta_2} T_1 e^{-\theta_1} T_1^* e^{\theta_2}) e^{-\theta_2} T_1 \tilde{\mathbf{u}}\|,$$

with $C'_N > 0$ independent of ε . Then, we have,

Lemma F.3. *The operator $J := e^{\theta_1-\theta_2} T_1 e^{-\theta_1} T_1^* e^{\theta_2}$ is bounded on $L^2(\mathbb{R}^{2n})$, uniformly with respect to $h > 0$ small enough.*

Proof. The distributional kernel of J is given by,

$$J(y, \eta; z, \zeta) = \int_{\mathbb{R}^n} e^{i(y-x)\eta/h + i(x-z)\zeta/h - (y-x)^2/2h - (x-z)^2/2h} \\ \times e^{\theta_2(\zeta) - \theta_2(\eta) + \theta_1(y_n) - \theta_1(x_n)} dx.$$

In this integral, we can perform the change of contour,

$$\mathbb{R}^n \ni x \mapsto x + ik^{2/3} \frac{\zeta - \eta}{|\zeta - \eta|},$$

and, since $|\nabla \theta_1| \leq \frac{1}{2}k^{2/3}$ and $|\nabla \theta_2| \leq \frac{1}{4}k^{2/3}$, we obtain,

$$|J(y, \eta; z, \zeta)| \leq \int_{\mathbb{R}^n} e^{-k^{2/3}|\zeta-\eta|/h - (y-x)^2/2h - (x-z)^2/2h + k^{4/3}/h} \\ \times e^{k^{2/3}|\zeta-\eta|/4h + k^{2/3}|y-x|/2h} dx.$$

Therefore, using the fact that $k^{2/3}|y-x| \leq 2k^{4/3} + (y-x)^2/2$ and $k^{4/3}/h \rightarrow 0$, for h small enough, we find,

$$|J(y, \eta; z, \zeta)| \leq 2 \int_{\mathbb{R}^n} e^{-3k^{2/3}|\zeta-\eta|/4h - (y-x)^2/4h - (x-z)^2/2h} dx,$$

and the result follows from the Schur Lemma (see, e.g., [Ma6]). \square

We deduce from Lemma F.3 and (F.7)-(F.8) that we have,

$$\begin{aligned} & \| |\eta|^{2N} e^{-\varepsilon k^{2/3}\langle \eta \rangle/h} T_1 \check{\mathbf{u}} \|_{\eta^2 \geq \max(C_1, 2|y_n|)} \\ & \leq C_N \| e^{-k\langle \eta \rangle/h} T_1 \tilde{\mathbf{u}} \| + C_N \| e^{-\varepsilon k^{2/3}\langle \eta \rangle/h} T_1 \check{\mathbf{u}} \|_{\eta^2 \leq C_1}, \end{aligned}$$

with C_1, C_N independent of ε , and thus,

$$\begin{aligned} & \| |\eta|^{2N} e^{-\varepsilon k^{2/3}\langle \eta \rangle/h} T_1 \check{\mathbf{u}} \|_{\eta^2 \geq \max(C_1, 2|y_n|)} \\ & \leq C_N \| e^{-k\langle \eta \rangle/h} T_1 \tilde{\mathbf{u}} \| + C_N h^{-C_1} \| e^{-k\langle \eta \rangle/h} T_1 e^{-4k\langle y_n \rangle^{2/3}} \tilde{\mathbf{u}} \|. \end{aligned}$$

Now, one can see as in the proof of Lemma F.3 that we have,

$$(F.9) \quad \| e^{-k\langle \eta \rangle/h} T_1 e^{-4k\langle y_n \rangle^{2/3}} \tilde{\mathbf{u}} \| \leq C \| e^{-k\langle \eta \rangle/h} T_1 \tilde{\mathbf{u}} \|,$$

and, in virtue of (6.32), we deduce,

$$\| |\eta|^{2N} e^{-\varepsilon k^{2/3} \langle \eta \rangle / h} T_1 \check{\mathbf{u}} \|_{\eta^2 \geq \max(C_1, 2|y_n|)} = \mathcal{O}(h^{-C_1}),$$

uniformly with respect to h and ε . Sending ε to 0 in this estimate, we conclude,

$$(F.10) \quad \| |\eta|^{2N} T_1 \check{\mathbf{u}} \|_{\eta^2 \geq \max(C_1, 2|y_n|)} = \mathcal{O}(h^{-C_1}).$$

Gathering (F.8), (F.9), (F.10), and using Lemma F.3, we obtain,

$$(F.11) \quad \| \langle \eta \rangle^{2N} T_1 \check{\mathbf{u}} \| = \mathcal{O}(h^{-C}),$$

and, inserting (F.11) into (F.6), and sending again ε to 0, this gives us,

$$(F.12) \quad \| |\eta|^{2N} e^{\psi/h} T_1 \check{\mathbf{u}} \|_{\text{Supp } \psi} = \mathcal{O}(h^{-C}).$$

Therefore, thanks to our choice of the function ψ , we finally obtain,

$$(F.13) \quad \| \langle \eta \rangle^{2N} T_1 \check{\mathbf{u}} \|_{K \times \{|\eta| \geq C_0\}} = \mathcal{O}(e^{-\delta_0/2h}).$$

where $K \subset \mathbb{R}^n$ is an arbitrary compact set, $C_0, N \geq 1$ are arbitrarily large and $\delta_0 = \delta_0(K, C_0, N)$ is a positive constant that does not depend on μ .

To complete the proof, we fix some arbitrary constant $C \geq 1$, we take $K' \subset\subset K \cap \{\langle y_n \rangle \leq C^2\}$, and we write,

$$\begin{aligned} & \| \langle \eta \rangle^{2N} T_1 \check{\mathbf{u}} \|_{K' \times \{|\eta| \geq 2C_0\}} \\ & \leq h^{-4C} \| \langle \eta \rangle^{2N} e^{-4k \langle y_n \rangle^{1/2}/h} T_1 \check{\mathbf{u}} \|_{K' \times \{|\eta| \geq 2C_0\}} \\ & \leq h^{-4C} \| \langle \eta \rangle^{2N} e^{-4k \langle y_n \rangle^{1/2}/h} T_1 e^{4k \langle y_n \rangle^{1/2}/h} \check{\mathbf{u}} \|_{K' \times \{|\eta| \geq 2C_0\}} \\ & \leq h^{-4C} \| J_1 (\langle \eta \rangle^{2N} T_1 \check{\mathbf{u}}) \|_{K' \times \{|\eta| \geq 2C_0\}}, \end{aligned}$$

with,

$$J_1 := \langle \eta \rangle^{2N} e^{-4k \langle y_n \rangle^{1/2}/h} T_1 e^{4k \langle y_n \rangle^{1/2}/h} T_1^* \langle \eta \rangle^{-2N}.$$

Then, in a similar way as in Lemma F.3, we see that the distributional kernel of J_1 satisfies,

$$|J_1(y, \eta; z, \zeta)| \leq 2 \int e^{-\delta |\zeta - \eta|/h - (y-x)^2/4h - (x-z)^2/2h + \delta^2/h} dx,$$

where $\delta > 0$ can be taken arbitrarily small. We deduce,

$$\begin{aligned} \| J_1 (\langle \eta \rangle^{2N} T_1 \check{\mathbf{u}}) \|_{K' \times \{|\eta| \geq 2C_0\}} & \leq 2e^{\delta^2/h} \| \langle \eta \rangle^{2N} T_1 \check{\mathbf{u}} \|_{K \times \{|\eta| \geq C_0\}} \\ & \quad + e^{\delta^2/h - \delta_1/h} \| \langle \eta \rangle^{2N} T_1 \check{\mathbf{u}} \|, \end{aligned}$$

where the positive number δ_1 only depends on the distance between K' and $\mathbb{R}^n \setminus K$. As a consequence, taking $\delta^2 \leq \min(\frac{1}{2}\delta_1, \frac{1}{4}\delta_0)$, and using (F.12)-(F.13) we conclude,

$$\|\langle \eta \rangle^{2N} T_1 \tilde{\mathbf{u}}\|_{K' \times \{|\eta| \geq 2C_0\}} = \mathcal{O}(h^{-4C} e^{-\delta_0/4h} + h^{-5C} e^{-\delta_1/2h}).$$

Since K was an arbitrary compact set, so is K' , and thus Lemma 6.9 is proved.

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